



TurbolEG.com



دفتر :

معادلات تفاضلية عادية 1

للطالبة : جود النسور

اللجنة الأكاديمية لقسم الهندسة الصناعية

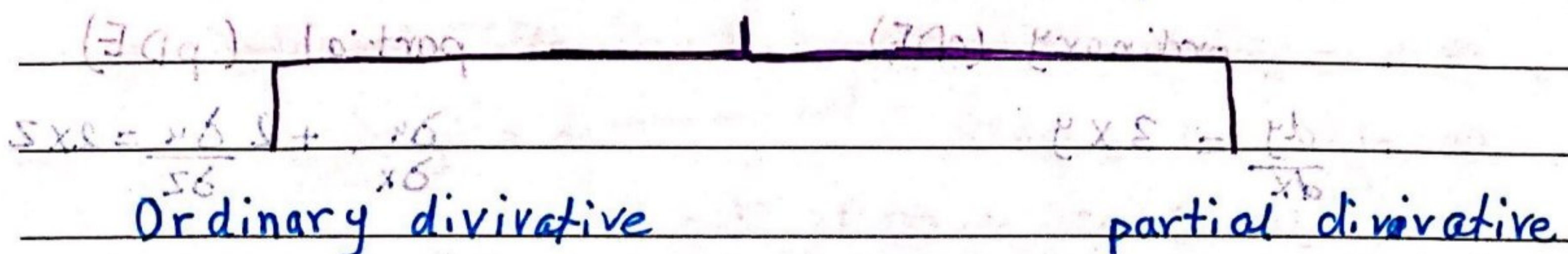
2023



Differential Equations Introduction

(7) Classification of differential equations

derivative (المشتقة)



المشتق العادي = (1)

المشتق الجزئي = (2)

Ex 8 $y = 3x^2 + \sin x$ — Ex 8 $u = 2xy - 3x^2 + y$

Find $y' = 6x + \cos x$ — Find $u_x = 2y - 6x$

Ex 9 $u = 2xy - 3x^2 + y$ — $u_x = \frac{\partial u}{\partial x} = 2y - 6x$

Ex 10 $u = 2xy - 3x^2 + y$ — $u_y = \frac{\partial u}{\partial y} = 2x + 1$

Ex 11 $u = 2xy - 3x^2 + y$ — $u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = 2$

Definition: An equation containing the derivatives of one or more dependent variable with respect to one or more independent variable is said to be a differential equation (DE)

كل معادلة تفاضلية (عادية أو جزئية) تكون (DE)

Ex 8 $y'' + y' = 5x$ — DE

Ex 9 $\frac{\partial u}{\partial x} + 2x = 4$ — DE

Ex 10 $\frac{\partial^2 u}{\partial x^2} + 2x = 4$ — DE

DE are classified by : type, order, linearity

الخاصة الرتبة النوع

الحظية

I classification by type :

العادية (ordinary)

ordinary (ODE)

$$\frac{dy}{dx} = 3xy$$

partial (PDE)

$$\frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial z} = 2xz$$

① $x^2 y' + y'' = 2x^2 \ln y \rightarrow$ ODE

② $\frac{d^2 y}{dx^2} + 5y = e^x \rightarrow$ ODE

③ $u_{xy} + 2u_x = 3x \rightarrow$ PDE

④ $\frac{\partial u}{\partial x} + 2x u = 3 \rightarrow$ PDE

⑤ $(y'')^2 + 5y' = 0 \rightarrow$ ODE

II Classification by the order :

Order of DE is the order of the highest derivative in the equation.

① $y'' - y' + y'' - 5x = 0 \rightarrow$ order = 2

② $(y^{(5)})'' - y^{(6)} - 2 \ln x = 7 \rightarrow$ order = 6

③ $(y^{(3)})^5 - 2y'' + y' = 12 \rightarrow$ order = 3

④ $\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} + 5 \rightarrow$ order = 2

III Classification as linear or non linear

An n^{th} order linear DE has the form

$$a_n^{(x)} y^{(n)} + a_{n-1}^{(x)} y^{(n-1)} + \dots + a_1^{(x)} y' + a_0^{(x)} y = g(x)$$

الـ y ومشتقاتها الأسس لها $\neq 1$ (بدون $\sin/\sqrt{\cdot}/\ln$ ليست في المقام)

معاملاتها (x) أو ثابتة $\rightarrow$ غير مفروبة يعنيها

الاقتران (y) يعتمد على (x) أو ثابت

$$0 = \frac{y}{x} - x^2$$

① $y'' + 2xy' - 3x^2y = e^x \rightarrow$ Linear in y المشتقة عليها

② $3r\theta' + 2\theta = r \rightarrow$ Linear in θ

③ $3yx'' + 2x''' = 2x \rightarrow$ Linear in x

④ $2xy'' + y' \cdot y^{(3)} + 2xy = 0 \rightarrow$ non linear

⑤ $2xy' + 2\sin(xy) = e^x \rightarrow$ non linear

⑥ $\frac{dy}{dx} = \frac{3}{x-e^y} \rightarrow$ non linear in y but linear in x

$$\frac{dx}{dy} = \frac{1}{3} \frac{x - e^y}{1} \Rightarrow \frac{dx}{dy} = \frac{x}{3} - \frac{e^y}{3}$$

$\frac{dx}{dy} = \frac{1}{3} x - \frac{e^y}{3} \Rightarrow \frac{dx}{dy} - \frac{1}{3} x = -\frac{e^y}{3} \rightarrow$ (Linear in x)

Solution of DE

على شكل اقتران $y = f(x)$ Explicit sol is $x = (y)$ implicit sol $x^2 + y^2 = 4$

* Show that $y = e^{2x}$ is an explicit sol to $y' - 2y = 0$

$$y' = 2e^{2x}$$

$$2e^{2x} - 2e^{2x} = 0 \quad \checkmark \quad \text{solution}$$

* Show that $4x^2 - y^2 = 4$ is an implicit sol to

$$y \cdot \frac{dy}{dx} - 4x = 0$$

$$8x - 2y \frac{dy}{dx} = 0$$

$$\div 2 \Rightarrow -4x + y \frac{dy}{dx} = 0 \quad \checkmark$$

$$-4x + y \frac{dy}{dx} = 0 \quad \checkmark$$

Remark: the ODE of n^{th} order has the form

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$$

Initial value problem (I.V.P)

"solve the following I.V.P: $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

(b) where $y(x_0) = y_0$, $y'(x_0) = y_1$, $y''(x_0) = y_2$

$$y^{(n-1)}(x_0) = y_{n-1}$$

Ex: show that $y(x)$ is a solution of the I.V.P

$$y' = x^2 + y^2 \quad \text{with } y(1) = 2 \quad \text{find } y'(1), y''(1), y'''(1)$$

$$y'(1) = 1^2 + 2^2 = 5$$

$$y''(1) = 2x + 2y y' = 2(1) + 2(2)(5) = 22$$

$$y'''(1) = 2 + 2(y'^2 + y y'') = 2 + 2(5^2 + 2 \cdot 22) = 112$$

$$y' = x^2 + y^2$$

$$(*) \quad y'(1) = (1)^2 + y^2(1) = 1 + (2)^2 = 5$$

$$(*) \quad y'' = 2x + 2y y'$$

$$y''(1) = 2 + 2 * 2 * 5 = 2 + 20 = 22$$

$$(*) \quad y''' = 2 + 2y y'' + 2y' y'$$

$$y'''(1) = 2 + 2 * 5 * 22 + 2 * 5 * 5 = 2 + 220 + 50 = 272$$

$$(*) \quad \frac{|x^2 - 0.5|}{p - 0.5} = \frac{p^2 + 1}{p^2 x}$$

$$\frac{|x^2 - 0.5|}{p - 0.5} = \frac{p^2 + 1}{p^2 x}$$

$$\frac{|x^2 - 0.5|}{p - 0.5} = \frac{p^2 + 1}{p^2 x}$$

$$x = 1$$

$$x = 1$$

$$0 < x - 0.5 < 0.5$$

$$x = 1$$

$$x = 1$$

Theorem Consider the following IVP

$$y' + p(x)y = g(x) \quad \text{where } y(x_0) = y_0$$

(*) if $p(x)$ & $g(x)$ are continuous on (a, b) that contains x_0 , then there is a unique solution for the IVP on interval (a, b)

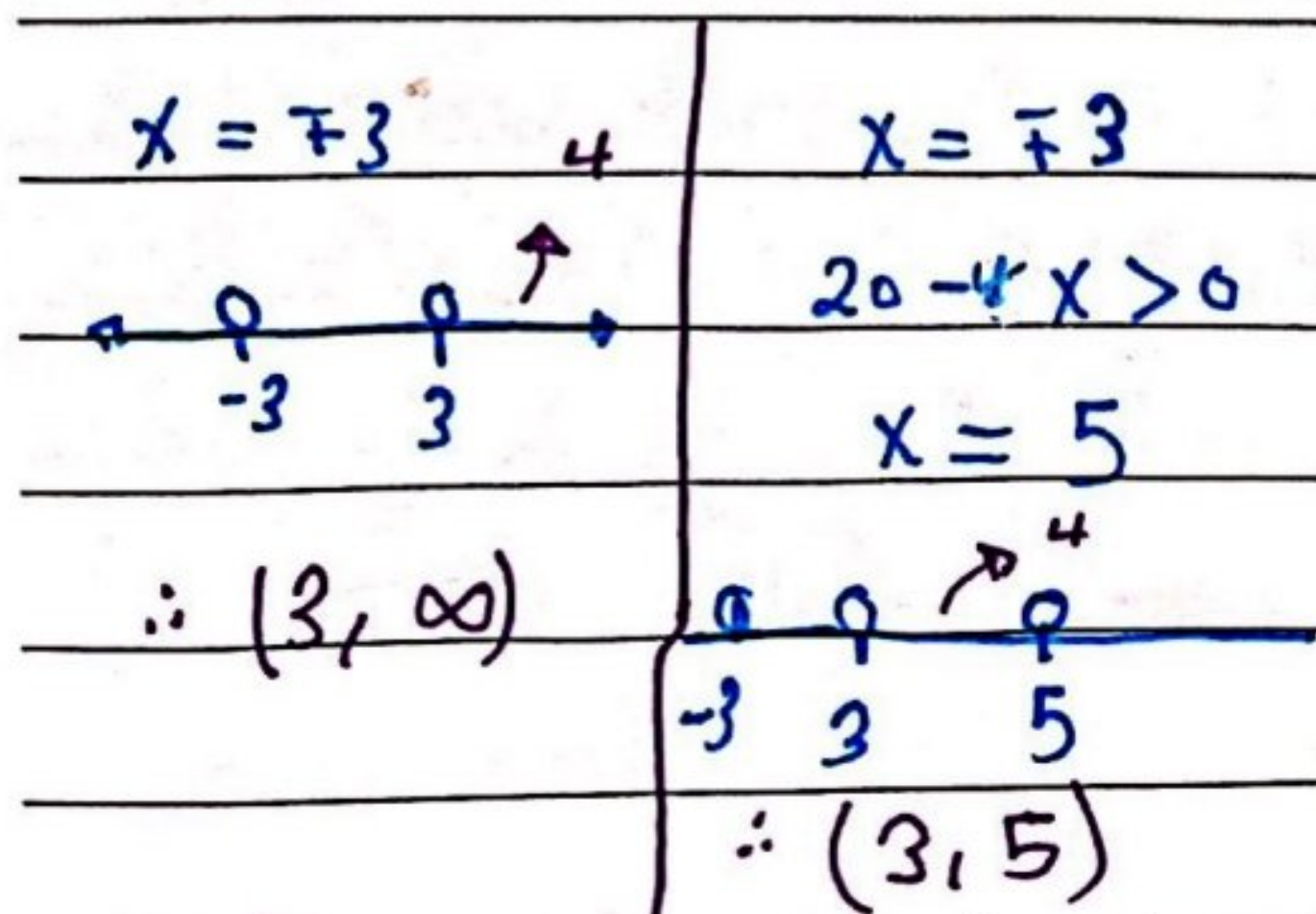
Ex: Find the largest open interval in which the following IVP has a unique solution

① $(x^2 - 9)y' + 2y = \ln |20 - 4x|$, $y(4) = -3$

$$y' + \frac{2y}{x^2 - 9} = \frac{\ln |20 - 4x|}{x^2 - 9}$$

$$p(x) = \frac{2}{x^2 - 9}$$

$$g(x) = \frac{\ln |20 - 4x|}{x^2 - 9}$$



يجب ان تكون مكتوبة بالشكل العام
 y' معاملها (1)

$$x_0 = 4$$

بوجه $p(x)$ و $g(x)$

بما قلنا نقاط الاتصال (المقادير)
 بخلاف الفترة التي فيها (x_0)
 نجد اننا طرح بينه الشرطين
 ايجز من صفر

$$(3, \infty) \cap (3, 5) = (3, 5)$$

$$(2) (x-9)y' + \frac{2x}{\ln(x-5)} y = \frac{3}{x}$$

$$y(7) = 3 \frac{\ln 2}{\ln 5} \quad (1)$$

$$y' + \frac{2x}{\ln(x-5)(x-9)} y = \frac{3}{x(x-9)}$$

$P(x) \qquad \qquad \qquad Q(x)$

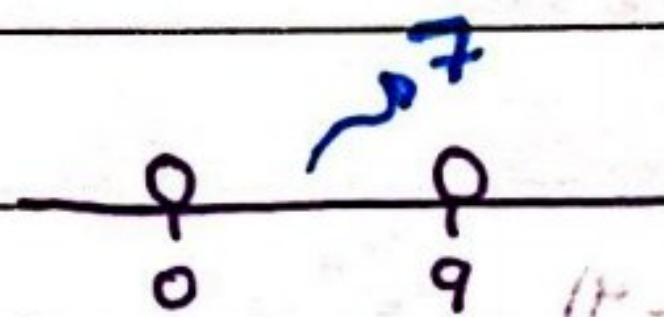
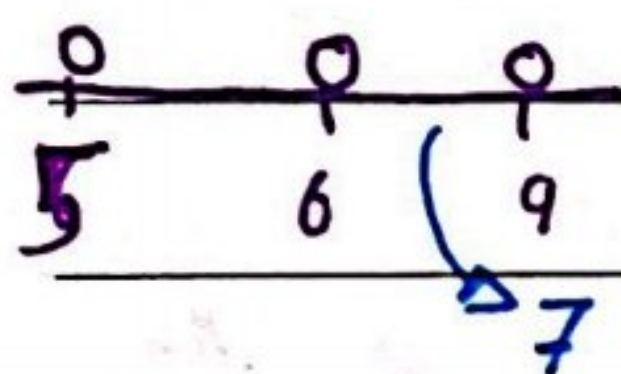
$$\frac{P + P'x}{x + xP} = \frac{P'Q}{xQ} \quad (2)$$

$$x=9 \quad x-5 > 0 \rightarrow x > 5 \quad x=0 \quad x=9$$

$$\frac{(1+x)P'}{(1+P')x}$$

$$\ln(x-5) = 0$$

$$x-5 = 1 \rightarrow x = 6$$



$$(1+x) \text{ viz } x = \frac{P'Q}{xQ} \quad (3)$$

$$\therefore (0,9)$$

$$\therefore (6,9) \text{ viz } P = \frac{P'Q}{xQ}$$

$$\text{then } (6,9) \cap (0,9) = (6,9) \text{ or}$$

Chapter 28 First order DE

The DE has the form $\frac{dy}{dx} = f(x,y)$ y, x in $\mathbb{R}$

Separable equation y is a function of x only

this DE has the form $\frac{dy}{dx} = g(x) \cdot h(y)$

Ex: Determine the separable Equation from the following

$$\textcircled{1} \frac{dy}{dx} = e^{x-3y}$$

$$e^x \cdot e^{-3y}$$

✓

$$\textcircled{2} \frac{dy}{dx} = \frac{xy+y}{yx+x}$$

$$\frac{y(x+1)}{x(y+1)}$$

$$= \frac{y}{y+1} \cdot \frac{x+1}{x}$$

$$\frac{x+1}{x}$$

$$\textcircled{3} \frac{dy}{dx} = x \sin(x+y)$$

not separable X

To solve the separable equation $\frac{dy}{dx} = g(x)h(y)$:

$$\textcircled{*} \frac{1}{h(y)} dy = g(x) dx$$

x/y free

$$\textcircled{*} \int \frac{1}{h(y)} dy = \int g(x) dx$$

بكال الترمين

Ex 2 Solve $(2+x) \frac{dy}{dx} = y+3$

$$(2+x) dy = (y+3) dx$$

$$\int \frac{1}{y+3} dy = \int \frac{1}{2+x} dx$$

$$\frac{1}{y+3} dy = \frac{1}{2+x} dx$$

$$\ln|y+3| = \ln|2+x| + C$$

implicit solution

$$e^{\ln |y+3|} = e^{\ln |2+x| + C}$$

$$|y+3| = |2+x| \cdot e^C$$

$$y+3 = \pm C |2+x|$$

$$y = -3 + C |2+x|$$

explicit solu

$$y = -3 - C |2+x|$$

explicit solu

Ex 8 solve

$$e^x y \frac{dy}{dx} = e^{-y-x} + e^{-2x-y}$$

$$e^{-x} y \frac{dy}{dx} = e^{-y-x} + e^{-3x-y}$$

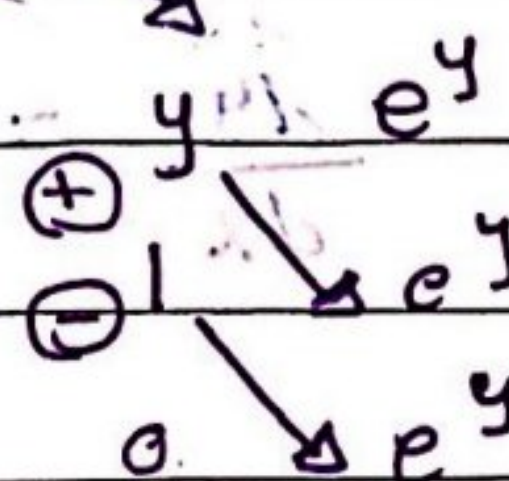
$$e^y y \frac{dy}{dx} = e^{-x} + e^{-3x}$$

$$y e^y dy = (e^{-x} + e^{-3x}) dx \rightarrow \text{separable}$$

$$\int y e^y dy = \int (e^{-x} + e^{-3x}) dx$$

$$y e^y - e^y = -e^{-x} - \frac{1}{3} e^{-3x} + C$$

implicit solution



Ex 8 Solve $\frac{dy}{dx} = \frac{xy + 3x + 2y + 6}{xy - 2y + 4y - 8}$

$$\frac{dy}{dx} = \frac{x(y+3) + 2(y+3)}{y(x-2) + 4(x-2)}$$

$$\frac{dy}{dx} = \frac{(y+3)(x+2)}{(x-2)(y+4)}$$

$$\frac{(y+4)}{(y+3)} dy = \frac{(x+2)}{(x-2)} dx \rightarrow \text{Separable}$$

$$\int \frac{y+4}{y+3} dy = \int \frac{x+2}{x-2} dx$$

$$y - \ln|y+3| = x + 4 \ln|x-2| + c \quad \text{implicit solution}$$

Ex 9 Find the value of m and n which make the DE is separable

$$\frac{dy}{dx} = \frac{xy + 2y + 3x + m^2}{xy - 2x + 4y - n}$$

$$\frac{y(x+2) + 3(x + \frac{m^2}{3})}{x(y-2) + 4(y - \frac{n}{4})}$$

$$x+2 = x + \frac{m^2}{3}$$

$$m^2 = 6$$

$$m = \pm\sqrt{6}$$

$$-2 = -\frac{n}{4}$$

$$n = 8$$

Exo solve the IVP $(x+1)dy - (y^2+1)dx = 0$ where $y(0)=1$

$$(x+1)dy - (y^2+1)dx = 0 \quad \text{where } y(0)=1$$

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x+1} \Rightarrow \text{separable}$$

$$\tan^{-1} y = \ln|x+1| + C$$

since $y(0)=1$
 $\downarrow \quad \downarrow$
 $x \quad y$

$$\tan^{-1}(1) = \ln|0+1| + C$$

$$C = \frac{\pi}{4}$$

$$\tan^{-1} y = \ln|x+1| + \frac{\pi}{4}$$

$\tan$
 $\downarrow$
 $y = \tan(\ln|x+1| + \frac{\pi}{4})$

$$\frac{dy}{dx} = \frac{y}{x} \quad (2)$$

W.W

$$(1) \quad x e^{-y} \sin x \, dx - y \, dy = 0$$

$$x e^{-y} \sin x \, dx = y \, dy$$

$$x \sin x \, dx = y e^y \, dy$$

$$\textcircled{4} \frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$$

Linear Equation

The standard form of this equation is

$$y' + p(x)y = q(x)$$

to solve this equation

(1) $p(x) = 1$

(2) Find the integrating factor as

$$I(x) = e^{\int p(x) dx}$$

(3) Multiply both sides of eq (1) by I(x)

$$I(x)y' + I(x)p(x)y = I(x)q(x)$$

Ex 8 If $f'(x) = \frac{f^2(x) + 1}{x + e}$ with $f(0) = 0$

Find $f(2)$

$$\frac{df}{dx} = \frac{f^2 + 1}{x + e}$$

$$\int \frac{df}{f^2 + 1} = \int \frac{dx}{x + e}$$

$$\tan^{-1} f = \ln|x + e| + C$$

$$\tan^{-1}(0) = \ln|e| + C$$

$$C = -1$$

$$\tan^{-1} f = \ln|x + e| - 1$$

$$f(x) = \tan(\ln|x + e| - 1)$$

$$f(2) = \tan(\ln|2 + e| - 1)$$

Linear Equation:

$$\frac{2-x-12+12}{2-x+12-12} = \frac{11}{x+10} \quad (1)$$

The standard form of this equation is:

Linear in y $\Rightarrow \underline{y'} + p(x)y = q(x)$

to solve this equation:

① $I = q'$ does

② Find the integrating factor as:

$$\mu(x) = e^{\int p(x) dx}$$

③ Multiply both sides of eq by $\mu(x)$, then we have $\mu(x) y' + \mu(x) p(x) y = \mu(x) q(x)$

$$\frac{d[\mu(x) y]}{dx} = \mu(x) q(x)$$

④ x $\downarrow$ $\frac{d[\mu(x) y]}{dx}$ $\rightarrow$ $\frac{d[\mu(x) y]}{dx} = \mu(x) q(x)$

$$\int \frac{d[\mu(x) y]}{dx} = \int \mu(x) q(x) dx$$

$$\mu(x) y = \int \mu(x) q(x) dx + C$$

$$y = \frac{\int \mu(x) q(x) dx + C}{\mu(x)}$$

Remark 8 ① $u'(x) = p(x) \cdot e^{\int p(x) dx} = p(1-x) u(x) = e^{\int p(x) dx}$
 $= p(x) u(x)$

② $p(x) = \frac{u'(x)}{u(x)} = \frac{x(1-x) - 1}{x}$

③ $\frac{d}{dx} [u(x) y] = u'(x) y + u(x) y' = (1-x) u(x) y + u(x) y'$

Ex: solve $(x^2+1) \frac{dy}{dx} + xy - x = 0$

$(x^2+1) y' + xy = x \rightarrow$ Linear in y

* $y' + \frac{xy}{(x^2+1)} = \frac{x}{x^2+1}$

* $p(x) = \frac{x}{x^2+1}$ $q(x) = \frac{x}{x^2+1}$

* $u(x) = e^{\int p(x) dx} = e^{\int \frac{x}{x^2+1} dx} = e^{\frac{1}{2} \ln(x^2+1)} = (x^2+1)^{\frac{1}{2}}$

* $y(x) = \frac{1}{u(x)} \int u(x) q(x) dx + \frac{C}{u(x)}$

$= \frac{1}{(x^2+1)^{\frac{1}{2}}} \int (x^2+1)^{\frac{1}{2}} \cdot \frac{x}{x^2+1} dx + \frac{C}{(x^2+1)^{\frac{1}{2}}}$

$= \frac{1}{(x^2+1)^{\frac{1}{2}}} \cdot (x^2+1)^{\frac{1}{2}} + \frac{C}{(x^2+1)^{\frac{1}{2}}} = 1 + \frac{C}{(x^2+1)^{\frac{1}{2}}}$

$$\textcircled{2} \quad xy' - (x-1)y = x^2$$

$$\textcircled{*} \quad y' - \frac{(x-1)y}{x} = x$$

$$\textcircled{*} \quad p(x) = \frac{-(x-1)}{x} = -1 + \frac{1}{x}$$

$$\textcircled{*} \quad \mu(x) = e^{\int p(x) dx} = e^{\int (-1 + \frac{1}{x}) dx}$$

integrating factor for linear equation

$$= e^{-x + \ln x} = e^{-x} \cdot e^{\ln x} = \boxed{xe^{-x}}$$

$$\textcircled{*} \quad y = \frac{1}{\mu(x)} \int \mu(x) q(x) dx + \frac{C}{\mu(x)}$$

$$= \frac{1}{xe^{-x}} \int x^2 e^{-x} dx + \frac{C}{xe^{-x}}$$

$$= \frac{1}{xe^{-x}} \left[-x^2 - 2x - 2 \right] e^{-x} + \frac{C}{xe^{-x}}$$

$$= -x - 2 - \frac{2}{x} + \frac{C}{xe^{-x}}$$

Solve IVP

$$y' - y = e^y (1+x) \quad (1)$$

$$\frac{dy}{dx} = \frac{1}{e^y - x} \quad \text{where } P(x,y) = 0, \quad Q(x,y) = e^y (1+x)$$

$$\frac{dx}{dy} = e^y - x = P(y) + Q(y)x$$

$$x' = e^y - x$$

$$x^2 + x = e^y \rightarrow \text{linear in } x$$

$$P(y) = 1, \quad Q(y) = e^y$$

$$\mu(y) = e^{\int P(y) dy} = e^{\int 1 dy} = e^y$$

$$x = \frac{1}{\mu y} \int \mu(y) Q(y) dy + \frac{C}{\mu y}$$

$$= \frac{1}{e^y} \int e^y e^y dy + \frac{C}{e^y}$$

$$x = \frac{1}{e^y} \cdot \frac{e^{2y}}{2} + \frac{C}{e^y} = \frac{e^y}{2} + \frac{C}{e^y}$$

$$1 = \frac{e^0}{2} + \frac{C}{e^0} \Rightarrow \boxed{C = \frac{1}{2}}$$

$$x = \frac{e^y}{2} + \frac{1}{2e^y} = \frac{e^y + e^{-y}}{2} = \cosh y$$

$$(3) \quad (x+2)^2 y' = 5 - 4y - 2xy$$

$$(x+2)^2 y' + 4y + 2xy = 5$$

$$(x+2)^2 y' + 2(x+2)y = 5x - 9$$

$$(*) \quad y' + \frac{2y}{x+2} = \frac{5}{(x+2)^2}$$

$$p(x) = \frac{2}{x+2}$$

$$q(x) = \frac{5}{(x+2)^2}$$

$$u(x) = e^{\int \frac{2}{x+2} dx} = e^{2 \ln(x+2)} = (x+2)^2$$

$$(*) \quad y = \frac{1}{(x+2)^2} \int (x+2)^2 * \frac{5}{(x+2)^2} dx + \frac{C}{(x+2)^2}$$

$$y = \frac{5x}{(x+2)^2} + \frac{C}{(x+2)^2} = \frac{5x+C}{(x+2)^2}$$

$$\frac{0}{1} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 = x$$

$$\frac{1-2}{1-2} = \frac{3}{2} + \frac{1}{2} = 2$$

$$\frac{2}{2} = \frac{1-3+2}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1$$

1. we solve differential eq of form $y' + p(x)y = q(x)$ 2T 8x3

Q1) $x \frac{dy}{dx} - 4y = x^6 e^x$ 3T 8x3

$$\boxed{1} \quad x \frac{dy}{dx} - 4y = x^6 e^x$$

$$\boxed{2} \quad \frac{dr}{d\theta} + r \sec \theta = \cos \theta$$

$$\boxed{3} \quad \frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$$\boxed{1} \quad \frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$$p(x) = -\frac{4}{x} \quad q(x) = x^5 e^x$$

$$\mu(x) = e^{\int -\frac{4}{x} dx}$$

$$y = \frac{1}{\mu(x)} \int \mu(x) q(x) dx + \frac{C}{\mu(x)}$$

$$\boxed{2} \quad p(\theta) = \sec \theta \quad q(\theta) = \cos \theta$$

$$\mu(\theta) = e^{\int \sec \theta d\theta}$$

$$r = \frac{1}{\mu(\theta)} \int \mu(\theta) q(\theta) d\theta + \frac{C}{\mu(\theta)}$$

Ex 8 If $\mu(x) = e^{\ln(x)}$ is an integrating factor.
 for the linear DE $xy' + 2x^2 p(x) y = x g(x)$
 find $p(x)$

$$y' + 2x p(x) y = g(x) \quad p(x) = \frac{\mu'}{\mu}$$

$$2x p(x) = \frac{\mu'}{\mu} = \frac{\frac{1}{x} \cdot e^{\ln x} \cdot \ln 2}{e^{\ln x}} =$$

$$2x p(x) = \frac{\ln 2}{x} \Rightarrow p(x) = \frac{1}{2x^2}$$

Bernoulli Equation

the standard form of this DE is

$$y' + p(x)y = f(x)y^n$$

note that ① if $n=0 \Rightarrow y' + p(x)y = f(x)$ Linear in y

② if $n=1 \Rightarrow y' + p(x)y = f(x)y$

$\Rightarrow y' + (p(x) - f(x))y = 0$ Linear in y

③ If $n \neq 0/1$, then the Bernoulli Equation become linear by use the transform $v = y^{1-n}$

Bernoulli $\rightarrow$ Linear

* multiply the Bernoulli equation by y^{-n} then we have

$$(y' + p(x)y = f(x)y^n) y^{-n}$$

$$y^{-n} y' + p(x) y^{1-n} = f(x)$$

* $v = y^{1-n}$

$$v' = (1-n) y^{-n} y' \Rightarrow \frac{v'}{1-n} = y^{-n} y'$$

* put this in $y^{-n} y' + p(x) y^{1-n} = f(x)$

$$\frac{v'}{1-n} + p(x)v = f(x)$$

$$(1-n)v' + (1-n)p(x)v = (1-n)f(x) \rightarrow \text{Linear in } v$$

~~$y' + p(x)y = f(x)y^n$~~ $\rightarrow$ Bernoulli in y
 $\Downarrow v = y^{1-n}$

$$v' + (1-n)v = (1-n)f(x) \Rightarrow \text{Linear in } v$$

Solve $\frac{y'}{x} + y = y^2$

$\frac{y'}{x} + xy = xy^2 \rightarrow$ Bernoulli in y

$n=2$
$v = y^{1-n} = y^{1-2}$

$p(x) = -x, q(x) = -x$

$\mu(x) = e^{\int p(x) dx} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$

$v = \frac{1}{\mu(x)} \int \mu(x) q(x) dx + \frac{C}{\mu(x)}$

$v = \frac{1}{e^{-\frac{x^2}{2}}} \left[\int -x e^{-\frac{x^2}{2}} dx + \frac{C}{e^{-\frac{x^2}{2}}} \right]$
 $= e^{\frac{x^2}{2}} [e^{-\frac{x^2}{2}}] + C e^{\frac{x^2}{2}}$

$v = 1 + C e^{\frac{x^2}{2}}$

but $v = y^{1-n} = y^{-1}$

$y^{-1} = 1 + C e^{\frac{x^2}{2}} \Rightarrow y = \frac{1}{1 + C e^{\frac{x^2}{2}}}$

$$\text{Solve } xy' + y = \ln x \cdot y^2$$

$$\cancel{y'} + \frac{\cancel{y}}{x} = \frac{\ln x}{x} \cancel{y^2} \rightarrow \text{Bernoulli in } y$$

$$v' - \frac{1}{x}v = -\frac{\ln x}{x} \rightarrow \text{linear in } v \quad \boxed{v = y^{1-n} = y^{-1}}$$

$$p(x) = \frac{-1}{x}$$

$$q(x) = \frac{\ln x}{x}$$

$$\mu(x) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

$$v = \frac{1}{\mu(x)} \int \mu(x) q(x) dx + \frac{C}{\mu(x)}$$

$$v = \frac{1}{\frac{1}{x}} \int \frac{1}{x} \cdot -\frac{\ln x}{x} dx + \frac{C}{\frac{1}{x}}$$

$$v = x \left[\frac{\ln x}{x} + \frac{1}{x} \right] + Cx$$

$$y^{-1} = \ln x + 1 + Cx \rightarrow \boxed{y = \frac{1}{\ln x + 1 + Cx}}$$

solve: $\frac{dy}{dx} = \frac{1}{\frac{x}{y} + yx^2}$

$$\frac{dx}{dy} = \frac{x}{y} + yx^2$$

$\frac{dx}{dy} - \frac{1}{y}x = yx^2 \rightarrow$ Bernoulli in $\frac{1}{x}$

$$v' + \frac{1}{y}v = -y$$

$n = 2 = (x)q$
 $v = x^{1-n} = x^{-1}$

$p(x) = \frac{1}{y} \quad q(x) = -y$

$\mu(x) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$$v = \frac{1}{y} \int -y^2 dy + \frac{C}{y}$$

$$= \frac{1}{y} \cdot \frac{-y^3}{3} + \frac{C}{y}$$

$$v = \frac{-y^2}{3} + \frac{C}{y} \Rightarrow x^{-1} = \frac{-y^2}{3} + \frac{C}{y}$$

Review about partial derivative

Exo: $p(x) = x^2 + xy + \sin(2x + 3y) - ye^{xy}$

① $p_x = \frac{\partial p}{\partial x} = 2x + y + 2\cos(2x + 3y) - 0 + ye^{xy}$

② $p_y = 0 + x + 3\cos(2x + 3y) - 1 + xe^{xy}$

Exo: ① $\int (xe^y + xy + \sin(x + 2y) + e^{xy}) dx$

$$\frac{x^2}{2} e^y + \frac{x^2}{2} y - \cos(x + 2y) + \frac{e^{xy}}{y} + g(y)$$

② $\int (xe^y + xy + \sin(x + 2y) + e^{xy}) dy$

$$= xe^y + \frac{y^2}{2} x - \frac{\cos(x + 2y)}{2} + \frac{e^{xy}}{x} + h(x)$$

Exact Equation

The DE $M(x, y) dx + N(x, y) dy = 0$

is said to be exact if $M_y = N_x$ $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right)$

determine whether the DE is exact or not

$$\boxed{1} \quad \underbrace{(5x^2 + 4y)}_M dx + \underbrace{(4x - 8y^3)}_N dy = 0$$

ببساطة بالـ y $\rightarrow$ M_y ببساطة بالـ x $\rightarrow$ N_x

$$M_y = 0 + 4 = 4 \quad N_x = 4 - 0 = 4 \Rightarrow \text{exact} \checkmark$$

$$P = (M)P$$

$$\boxed{2} \quad \underbrace{(\sin y - x^2 \cos y)}_M dx + \underbrace{(x \cos y - \sin y)}_N dy = 0$$

$$M_y = \cos y + x \sin y \neq N_x = \cos y \Rightarrow \text{Not exact}$$

Remark: if $F(x,y) = C$ is a solution of the **exact**

$$\text{DE } M dx + N dy = 0 \quad \text{then } \frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

Exo Solve: $\underbrace{(\cos x \sin x - xy^2)}_M dx + \underbrace{(y - yx^2)}_N dy = 0$

$$M = -2xy \quad N = -2yx \Rightarrow \text{exact} \checkmark$$

Suppose the solution is $F(x,y) = C$ such that

$$\frac{\partial F}{\partial x} = M \quad \frac{\partial F}{\partial y} = N$$

$$\frac{\partial F}{\partial x} = \cos x \sin x - xy^2 \quad \frac{\partial F}{\partial y} = y - yx^2$$

$$* \int \frac{\partial F}{\partial x} dx = \int (\cos x, \sin x - xy^2) dx + g(y)$$

$$F(x, y) = -\frac{\cos(2x)}{4} - \frac{x^2 y^2}{2} + g(y)$$

$$* \frac{\partial F}{\partial y} = 0 - 2xy + g'(y) = y - yx^2$$

$$g'(y) = y$$

$$g(y) = \int y dy = \frac{y^2}{2}$$

$$x \text{ times: } F(x, y) = -\frac{\cos(2x)}{4} - \frac{x^2 y^2}{2} + \frac{y^2}{2}$$

$$\text{the solution is } -\frac{\cos(2x)}{4} - \frac{x^2 y^2}{2} + \frac{y^2}{2} = C$$

Remark ① If $M dx + N dy = 0$ is exact then the

$$\text{solution is: } \int M dx + \int_{\text{الحدود التي لا يوجد فيها } N} dy = C$$

② If $M dx + N dy = 0$ is exact eq then

$$\text{the solution is: } \int_{\text{الحدود التي لا يوجد فيها } M} dx + \int N dy = C$$

Ex 8 Solve: $\underbrace{(\cos x \sin x - xy^2)}_M dx + \underbrace{(y - yx^2)}_N dy = 0$

Solu: $M_y = -2xy$

$N_x = -2xy$

Since $M_y = N_x \Rightarrow$ exact

$$\int M dx + \int N dy = C$$

$$\int (\cos x \sin x - xy^2) dx + \int y dy = C$$

$$\frac{-\cos(2x)}{4} - \frac{x^2 y^2}{2} + \frac{y^2}{2} = C$$

Ex 8 Solve: $(ye^{xy} + y^2 - \frac{y}{x^2} + x) dx + (xe^{xy} + 2xy) dy = (3 - \frac{1}{x}) dx$

Solu:

$$\underbrace{(ye^{xy} + y^2 - \frac{y}{x^2} + x)}_M dx + \underbrace{(xe^{xy} + 2xy)}_N dy = (3 - \frac{1}{x}) dx$$

$$M_y = e^{xy} + yxe^{xy} + 2y - \frac{1}{x^2}$$

$$M_x = e^{xy} + yxe^{xy} + 2y - \frac{1}{x^2}$$

exact ✓

Solution: $\int ye^{xy} + y^2 - \frac{y}{x^2} + x dx + \int -3 dy = C$

$$e^{xy} + y^2 x + \frac{y}{x} + \frac{x^2}{2} - 3y = C$$

Ex 8 Find the value of a and b which make the following DE is exact.

$$(xy^3 + by^2)dx + (ay^2x^2 + xy + y)dy = 0$$

$$\begin{matrix} M_y & N_x \\ 3xy^2 + 2by & = 2ay^2x + 1 \end{matrix}$$

$$\begin{aligned} 2b &= 1 \Rightarrow b = \frac{1}{2} \\ 3x &= 2ax \Rightarrow a = \frac{3}{2} \end{aligned}$$

Ex 9 If $\cos(xy) - 2x + \ln y = e^x - 3y^2 + 5$

is a solution for an exact DE

$$Mdx + Ndy = 0$$

Find the functions $M(x, y)$ $N(x, y)$?

8 مشتقة F بالمتغير

M هي x

① متابع المتغيرة x

متغيراته

$$\cos(xy) - 2x + \ln y = e^x - 3y^2 + 5 \Rightarrow f(x, y)$$

8 مشتقة F بالمتغير

N هي y

② مشتقة F بالمتغير y و x متغير بالثابت

$$\otimes M = -y \sin(xy) - 2 - e^x$$

$$\otimes N = -x \sin(xy) + \frac{1}{y} + 6y$$

Remark: If $Mdx + Ndy = 0$ is exact eq then

$$① \quad M(x,y) = \int N_x dy - h(x)$$

إذا كانت N متغيرة x بالدرجة الأولى N_x بالدرجة الأولى $h(x)$ بالدرجة الأولى

$$② \quad N(x,y) = \int M_y dx + g(y)$$

إذا كانت M متغيرة y بالدرجة الأولى M_y بالدرجة الأولى $g(y)$ بالدرجة الأولى

Exact 8

$$M_y = N_x$$

Ex: Find the general function $N(x,y)$ so that

$$\left(y e^{xy} + y^2 - \frac{y}{x^2} \right) dx + N(x,y) dy = 0 \text{ is exact}$$

$$N(x,y) = \int M_y dx + g(y)$$

$$= \int e^{xy} + x y e^{xy} + 2y - \frac{1}{x^2} + g(y)$$

$$N(x,y) = x e^{xy} + 2yx + \frac{1}{x} + g(y)$$

إذا كانت N متغيرة x بالدرجة الأولى N_x بالدرجة الأولى

* Where $N(1,y) = 5$

$$N(1,y) = e^y + 2y + 1 + g(y) = 5$$

$$g(y) = 4 - e^y - 2y$$

إذا ما قيمت هذه
الدرجة الأولى في
معادلة حتم لو افعلت
 $g(y)$

Ex 8. Find the general function $u(x, y)$ so that the following DE is exact:

$$\underbrace{(4u(x, y) + 2xy)}_M dx + \underbrace{(\sec^2 y + \frac{x}{y})}_N dy = 0$$

$$\frac{\partial}{\partial y} (4u(x, y) + 2xy) = N_x = \frac{1}{y}$$

$$\frac{\partial}{\partial y} (4u(x, y) + 2xy) = \frac{1}{y}$$

$$\int \frac{\partial}{\partial y} (4u(x, y) + 2xy) dy = \int \frac{1}{y} dy + g(x)$$

$$4u(x, y) + 2xy = 2 \ln |y| + g(x)$$

$$u(x, y) = \frac{-2xy + 2 \ln |y| + g(x)}{4}$$

H.w 8. ① Find the value of a and b so that the DE is exact:

$$(ay^3 + bxy^4 - 2x) dx + (3xy^2 + 20x^2y^3) dy = 0$$

Special Integrating factor

If the DE $Mdx + Ndy = 0$ (*) is not exact then

① If the function $\frac{My - Nx}{N}$ depends only on x

then the special integrating factor for eq (*) is :

$$\mu(x) = e^{\int \frac{My - Nx}{N} dx}$$

DE (*) becomes exact if it is multiplied by $\mu(x)$

② If the function $\frac{Nx - My}{M}$ depends on y only

then the special integrating factor of DE (*)

is : $\mu(y) = e^{\int \frac{Nx - My}{M} dy}$

DE (*) becomes exact if its multiplied by $\mu(y)$

Ex: Find the integrating factor for the following non-exact DE.

$$\boxed{1} \quad (x+y) dx + x \ln x dy = 0$$

$$M_y = 1$$

$$N_x = \ln x + 1 \cdot \frac{1}{x} = \frac{1 + \ln x}{x}$$

$M_y \neq N_x$ Not exact

$$\frac{M_y - N_x}{N} = \frac{1 - \ln x - 1}{x \ln x} = -\frac{\ln x}{x \ln x} = -\frac{1}{x}$$

the special integrating factor is $\mu(x) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$$\mu(x) = e^{\int \frac{-1}{x} dx}$$

$$= e^{-\ln x}$$

$$= x^{-1} = \frac{1}{x}$$

$$\boxed{2} \quad 6xy dx + (4y + 9x^2) dy = 0$$

$$M_y = 6x$$

$$N_x = 18x \neq 6x \Rightarrow \text{Not exact}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{\int \frac{2}{y} dy} = e^{2 \ln y} = y^2$$

$$= y^2$$

Solve the IVP

$$\underbrace{y dx}_M + \underbrace{(2x - ye^y) dy}_N = 0$$

where $y(1) = 0$

$$M_y = 1 \neq N_x = 2 \Rightarrow \text{not exact}$$

$$\frac{N_x - M_y}{M} = \frac{2 - 1}{y} = \frac{1}{y}$$

$$M(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

multiply the DE by $M(y) = y$

$$\underbrace{y^2 dx}_M + \underbrace{(2xy - y^2 e^y) dy}_N = 0 \Rightarrow \text{exact}$$

$$\int M dx + \int N dy = C$$

$$\int y^2 dx + \int -y^2 e^y dy = (Cx + P) + Q = C$$

$$y^2 x + [-y^2 + 2y - 2]e^y = C$$

$$y(1) = 0$$

$$0^2(1) + (-0^2 + 2(0) - 2)e^0 = C$$

$$\boxed{C = -2}$$

$$y^2 x + (-y^2 + 2y - 2)e^y = -2$$

Find solution of $y dx + (y^2 - x) dy = 0$
 $M_y = 1 \neq N_x = -1 \Rightarrow M_y \neq N_x \Rightarrow$ Not exact

$$\frac{M_x - N_y}{N} = \frac{-1 - 1}{y} = -\frac{2}{y} \quad \text{y dependent}$$

$$\mu(y) = e^{\int -\frac{2}{y} dy} = e^{-2 \ln y} = y^{-2} = \frac{1}{y^2}$$

Multiply the DE by $\mu(y)$

$$\frac{1}{y} dx + \left(1 - \frac{x}{y^2}\right) dy = 0 \quad \text{exact}$$

$$\int M dx + \int N dy = C \quad \left(\int \frac{1}{y} dx = x/y \right)$$

$$\int \frac{1}{y} dx + \int 1 dy = C$$

$$\frac{x}{y} + y = C$$

If $\mu y = y^k$ is an integrating factor for the non-exact equation

$$y^k (xy dx + (2x^2 + 3y^2 - 20) dy) = 0$$

Find k

$$x y^{k+1} dx + (2x^2 y^k + 3y^{2+k} - 20y^k) dy = 0 \rightarrow \text{exact}$$

$$\begin{aligned} \mu_y &= N_x \\ (k+1)xy^k &= 4xy^k \end{aligned}$$

$$k+1=4$$

$$\boxed{k=3}$$

Ex8 The CDE $(x^2 + y) dx + F(x) dy = 0$ becomes exact by the integrating factor $\mu(x) = x$

Find $F(x)$

Multiply by $\mu(x) = x$

$$(x^3 + xy) dx + x F(x) dy = 0$$

$$M(y) = N(x)$$

$$x = \frac{\partial (x F(x))}{\partial x}$$

$$\int x dx = \int \frac{\partial (x F(x))}{\partial x}$$

$$\frac{1}{2} x^2 + [g(y)] = x F(x) \Rightarrow \frac{1}{2} x^2 + C = x F(x)$$

كتابة (C) لأنه ناتج عن تكامل (A)

وهو ناتج على y

$$F(x) = \frac{x}{2} + \frac{C}{x}$$

(*) Remarks If $\mu(x,y) = x^p y^q$ is an integrating factor for the non-exact DE

$$M dx + N dy = 0$$

then we use the condition $\mu M_x - N_y = p \frac{N}{x} - q \frac{M}{y}$ to find p and q

Find the integrating factor $M(x, y) = x^p y^q$

for $(4xy^2 + 3y) dx + (3x^2y + 2x) dy = 0$

$$My - Nx = p \frac{N}{x} - q \frac{M}{y}$$

$$8xy + 3 - (6xy + 2) = p \cdot \frac{(3x^2y + 2x)}{x} - q \cdot \frac{(4xy^2 + 3y)}{y}$$

$$2xy + 1 = p(3xy + 2) - q(4xy + 3)$$

$$2xy + 1 = 3pxy + 2p - 4qxy - 3q$$

$$\underbrace{2xy + 1}_{\text{y قابل } \bar{y}^0} = \underbrace{(3p - 4q)xy}_{\text{y قابل } \bar{y}^1} + \underbrace{(2p - 3q)}_{\text{constant}} \quad \text{equation polynomial at level}$$

$$\boxed{2 = 3p - 4q} \quad (1)$$

$$\boxed{1 = 2p - 3q} \quad (2)$$

$$\boxed{p = 2}$$

$$\boxed{q = 1}$$

$$M(x, y) = x^p y^q = x^2 y$$

Now find the integrating factor of the form

$M(x, y) = x^p y^q$ which makes the DE exact

$$(12 + 5xy) dx + (6xy^2 + 3x^2) dy = 0$$

Homogeneous Equation :

Def: A function $G(x, y)$ is said to be homogeneous of order n if $G(tx, ty) = t^n G(x, y)$

إذا كان $G(x, y)$ متجانساً من الدرجة n فإن $G(tx, ty) = t^n G(x, y)$ (*)

$$0 = \mu \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) G(x, y) = \mu \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (x^2 + 2xy) = 2\mu x + 2\mu y = 2\mu(x+y)$$

Ex: $G(x, y) = x^2 + 2xy$ Is G homogeneous?

$$G(tx, ty) = (tx)^2 + 2(tx)(ty)$$

$$t^2 x^2 + 2 t^2 xy$$

$$t^2 (x^2 + 2xy) = G(x, y) \text{ homog with order 2}$$

Ex: $G(x, y) = x^2 + y^2 + 2xy^2$ Is G homogeneous?

$$G(tx, ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)^2$$

$$= t^2 x^2 + t^2 y^2 + 2 t^3 xy^2$$

$$= t^2 (x^2 + y^2 + 2t xy^2) \neq G(x, y) \text{ not homog}$$

Def: The DE $M(x, y) dx + N(x, y) dy = 0$

is said to be homogeneous if $M(x, y)$ and $N(x, y)$

are homog of same order

Ex: $(x^2 + y^2) dx + (yx + 2y^2) dy = 0$ Is DE homog?
 homog. of order 2

$$(x^2 + xy) dx + (2xy + y^2 + 5) dy = 0$$

not homog. because of the constant 5

$$\frac{dy}{dx} = \frac{4x^3 - xy^2}{2y^3 + x^2y}$$

homog. of order 3

to solve the homog. eq $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ we use the substitution $v = \frac{y}{x}$ to transform the homog into separable

Ex: solve $x^2 dy - (x^2 + xy + y^2) dx = 0$

$$x^2 dy = (x^2 + xy + y^2) dx$$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \Rightarrow \text{homog}$$

$$= \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

⊗ take $v = \frac{y}{x} \Rightarrow xv = y$

$$v + x \frac{dv}{dx} = \frac{dy}{dx}$$

$$\cancel{v} + x \frac{dv}{dx} = 1 + \cancel{v} + v^2$$

$$x \frac{dv}{dx} = 1 + v^2$$

$$\int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx$$

separable ✓

$$\tan^{-1} v = \ln|x| + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C$$

$$y = x \tan(\ln|x| + C)$$

Ex 8 Solve $\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x}$ $\Rightarrow$ homogeneous

$$\frac{dy}{dx} = \frac{2y}{2x} + \frac{1}{2} \frac{\sqrt{x^2 - y^2}}{x} = \frac{y}{x} + \frac{1}{2} \sqrt{\frac{x^2 - y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2} \sqrt{1 - \left(\frac{y}{x}\right)^2} \Rightarrow v = \left(\frac{y}{x}\right) \Rightarrow xv = y \Rightarrow v + x \frac{dv}{dx} = \frac{dy}{dx}$$

$$\cancel{v} + x \frac{dv}{dx} = \cancel{v} + \frac{1}{2} \sqrt{1 - v^2}$$

$$\int \frac{1}{\sqrt{1-v^2}} dv = \int \frac{1}{2x} dx \quad (\text{Separable})$$

$$\sin^{-1} v = \frac{1}{2} \ln |x| + C = \frac{1}{2} \ln x$$

$$\sin^{-1} \left(\frac{y}{x} \right) = \frac{1}{2} \ln |x| + C$$

$$y = x \sin \left(\frac{1}{2} \ln |x| + C \right)$$

Solve: $x dy - y (\ln y - \ln x + 1) dx = 0$

$$x dy = (y \ln \left[\frac{y}{x} \right] + y) dx \Rightarrow \text{homog}$$

$$\frac{dy}{dx} = \frac{y \ln \frac{y}{x} + y}{x}$$

$$\frac{dy}{dx} = \frac{y \ln \frac{y}{x} + y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x} + \frac{y}{x} \quad \text{homog}$$

take $v = \frac{y}{x}$ $xv = y$ $v + x \frac{dv}{dx} = \frac{dy}{dx}$

$$v + x \frac{dv}{dx} = v \ln v + v$$

$$\int \frac{1}{v \ln v} dv = \int \frac{1}{x} dx \quad \text{separable}$$

$$\ln |\ln v| = \ln |x| + C$$

$$\ln \left| \ln \left(\frac{y}{x} \right) \right| = \ln |x| + C$$

$$\ln \left(\frac{y}{x} \right) = C |x|$$

Ex: Find the value of k and m so that the xDE is homog.

$$(3\sqrt{xy^3} - 3m + 4)dx + x^2 y^k dy = 0$$

Solve: $(3x^{\frac{1}{2}}y^{\frac{3}{2}} - 3m + 4)dx + x^2 y^k dy = 0$

homog $\Rightarrow \textcircled{*} \frac{1}{2} + \frac{3}{2} = 2 + k$

$$k = 0$$

and $\textcircled{*} -3m + 4 = 0 \Rightarrow m = \frac{4}{3}$

k.w: solve $2\sqrt{xy} - y dx - x dy = 0$

Equation of the form $\frac{dy}{dx} = G(ax+by+c)$

This DE can be reduced to separable equation by use the substitution $v = ax+by+c$

Ex: solve $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

$v = y - 2x + 3$

$y = v + 2x - 3$

$\frac{dy}{dx} = \frac{dv}{dx} + 2$

$\frac{dv}{dx} + 2 = 2 + \sqrt{v}$

$\int \frac{dv}{\sqrt{v}} = \int dx \rightarrow \text{separable}$

$\sqrt{v} = \frac{1}{2}x + C$

$\frac{1}{2}x + C = \sqrt{v}$

$\sqrt{v} = \frac{1}{2}x + C$

$v = (\frac{1}{2}x + C)^2 \Rightarrow y - 2x + 3 = (\frac{1}{2}x + C)^2$

$y = (\frac{1}{2}x + C)^2 + 2x - 3$

Ex 8 solve $\frac{dy}{dx} = \sin(x-y)$

$v = x - y$

$y = x - v$

$\frac{dy}{dx} = 1 - \frac{dv}{dx} \Rightarrow 1 - \frac{dv}{dx} = \sin v$

$\int \frac{dv}{1 - \sin v} = \int dx \Rightarrow \tan v + \sec v = x + C$

$\tan(x-y) + \sec(x-y) = x + C$

Exemples 8 Solve 8

$\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$

$\frac{dy}{dx} = y - x - 1 + \frac{1}{x - y + 2} = y - x - 1 + \frac{1}{-2 - (y - x)}$

$v = y - x \Rightarrow y = v + x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} + 1$

$\frac{dv}{dx} + 1 = v + x - 1 + \frac{1}{-v + 2} \Rightarrow \frac{dv}{dx} = v - 2 + \frac{1}{2 - v}$

$\frac{dv}{dx} = \frac{(v-2)^2 - 1}{v-2} = \frac{v^2 - 4v + 4 - 1}{v-2} = \frac{v^2 - 4v + 3}{v-2}$

$$\int \frac{v-2}{v^2-4v+3} dv = \int dx \Rightarrow \frac{1}{2} \ln |v^2-4v+3| = \frac{x}{x^0} + C$$

$$\frac{1}{2} \ln |(y-x)^2 - 4(y-x) + 3| = \frac{x}{x^0} + C$$

How to solve

$$\frac{dy}{dx} = \left(\frac{1}{3x+y} \right)^5 - 3$$

$$\frac{1}{x} = \frac{(1/x) \cdot x + (1/x) \cdot y}{(1/x) \cdot x + (1/x) \cdot y}$$

The substitution $v = x^2 y$ is used to solve the ODE

$$x \frac{dy}{dx} + y = x^2 y + \frac{1}{x}$$

$$\frac{1}{x} \frac{dy}{dx} + \frac{y}{x} = x y + \frac{1}{x^2} \Rightarrow \frac{d}{dx} \left(\frac{y}{x} \right) = x \left(\frac{y}{x} \right) + \frac{1}{x^2}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = x \left(\frac{y}{x} \right) + \frac{1}{x^2}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = x \left(\frac{y}{x} \right) + \frac{1}{x^2}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = x \left(\frac{y}{x} \right) + \frac{1}{x^2}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = x \left(\frac{y}{x} \right) + \frac{1}{x^2}$$

Solve $\frac{dy}{dx} = \frac{2y}{x} + \cos\left(\frac{y}{x^2}\right)$

$v = \frac{y}{x^2} \Rightarrow y = x^2 v \Rightarrow \frac{dy}{dx} = 2xv + x^2 \frac{dv}{dx}$

$\cancel{2xv} + x^2 \frac{dv}{dx} = \cancel{\frac{2x^2 v}{x}} + \cos v$

$x^2 \frac{dv}{dx} = \cos v \Rightarrow \int \frac{1}{\cos v} dv = \int \frac{1}{x^2} dx$

$= \ln |\sec v + \tan v| = \frac{x^{-1}}{-1} + c$

$\ln \left| \sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right) \right| = c - \frac{1}{x}$

The substitution $v = x^2 \ln y$ transforms the DE

$\underline{x^2 y} + 4x^2 y \ln y = y \sin x$ into linear in v

$x^2 \ln y = v \Rightarrow v' = 2x \ln y + \frac{x^2}{y} y'$
 $\boxed{\ln y = \frac{v}{x^2}}$

$y v' = 2xy \ln y + x^2 y \Rightarrow y v' - 2xy \ln y = \underline{x^2 y'}$

$y v' - 2xy \ln y + 4x y^2 \ln y = y \sin x$

$v' - 2x \ln y + 4x^2 \ln y = \sin x$

$v' - 2x \frac{v}{x^2} + 4x^2 \frac{v}{x^2} = \sin x$

$v' + \left(4 - \frac{2}{x}\right)v = \sin x \rightarrow \text{Linear in } v$

classify each D.E. as separable, linear, exact, homogeneous Bernoulli &

$$\textcircled{1} \frac{dy}{dx} = \frac{x-y}{x}$$

* Linear, because $\frac{dy}{dx} = \frac{x}{x} - \frac{y}{x} \Rightarrow y' + \frac{1}{x}y = 1$

* homogeneous because $\frac{dy}{dx} = \frac{x}{x} - \frac{y}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{y}{x}$

* exact because $\frac{dy}{dx} = \frac{x-y}{x} \Rightarrow x dy = (x-y) dx$
 $(y-x) dx + x dy = 0$

$M y' = N \Rightarrow N x = 1 \checkmark$ exact

$$\textcircled{2} \frac{dy}{dx} = \frac{1}{y-x}$$

* $\frac{dx}{dy} = y-x \Rightarrow x' + x = y$ Linear in x

* take $v = y-x$

$$\textcircled{3} (x+1) \frac{dy}{dx} = -y+10$$

* Linear in y $(x+1)y' = -y+10 \Rightarrow (x+1)y' + y = 10$

* exact $(x+1) dy = (-y+10) dx \Rightarrow \underbrace{(x+1)}_M dy + \underbrace{(y+10)}_N dx = 0$

* separable $\frac{1}{-y+10} dy = \frac{1}{x+1} dx$

Linear second and higher order DEs

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x) \quad (1)$$

Remark 8 (1) If $g(x) \neq 0$ the eq (1) is called the non-homogeneous, but if $g(x) = 0$ the eq (1) is called homog

Ex 8 $x^2 y'' + 3xy' - 4y = 0$ homog

$x(y'') + 2xy' - 5y = e^x$ non-homog

(2) If $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ are constant then $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$ is called Linear DE with constant coefficients.

Ex 8 $3y'' - 2y' + 5y = 0$

Theorem 8 Superposition principle

If $y_1, y_2, \dots, y_n$ are solutions for

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$$

then $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ is a solution

Def 3 If $y_1, y_2, \dots, y_n$ are diff. functions of $(n-1)$ order, then the Wronskian of $y_1, y_2, \dots, y_n$ is given as

$$W[y_1, y_2, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1^{(2)} & y_2^{(2)} & \dots & y_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Ex 8 Find

$$W[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = a \cdot d - b \cdot c = 1 \cdot 2x - x^2 \cdot 1 = 2x - x^2$$

نشت وقادحة

$$= x \cdot 2x - x^2 = 2x^2 - x^2 = x^2$$

لا افتقانة - نشت حنين

$$W[\cosh x, \sinh x] = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix}$$

$$= \cosh^2 x - \sinh^2 x = 1$$

$$\textcircled{3} \quad W[\cos 2x, \sin 2x] = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x = 2$$

$$\textcircled{4} \quad W[x^2, x^3, \cos x] = \begin{vmatrix} x^2 & x^3 & \cos x \\ 2x & 3x^2 & -\sin x \\ 2 & 6x & -\cos x \end{vmatrix}$$

لثبات كسبت

$$= -3x^4 \cos x + -2x^3 \sin x + 12x^2 \cos x$$

$$= (6x^2 \cos x + -6x^3 \sin x + -2x^4 \cos x)$$

$$= (-5x^4 + 6x^2) \cos x + 4x^3 \sin x$$

$$\textcircled{5} \quad W[2, x, x^2, x^3, e^x] = \begin{vmatrix} 2 & x & x^2 & x^3 & e^x \\ 0 & 1 & 2x & 3x^2 & e^x \\ 0 & 0 & 2 & 6x & e^x \\ 0 & 0 & 0 & 6 & e^x \\ 0 & 0 & 0 & 0 & e^x \end{vmatrix}$$

المصفوفة التي فيها جزء صف اصفار تكون صفها صفر
لذلك لا نحتاج الى حساب ضرب العناصر على الصفوف

$$2 * 1 * 2 * 6 * e^x = 24e^x$$

Def 8: we say that $y_1, y_2, \dots, y_n$ are linearly independent on I if $W(y_1, y_2, \dots, y_n) \neq 0 \quad \forall x \in I$

otherwise are dependent $\{y_1, y_2, \dots, y_n\} \Rightarrow 0$

Ex 8 show that $\{2, e^x\}$ are independent on $(-\infty, \infty)$

$$W[2, e^x] = \begin{vmatrix} 2 & e^x \\ 0 & e^x \end{vmatrix} = e^x \cdot 2 - 0 = 2e^x \neq 0 \quad \forall x \in (-\infty, \infty)$$

$\therefore \{2, e^x\}$ are linearly independent

Ex 8 IF $\{x, x^2\}$ are independent on $(-\infty, \infty)$

$$\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

لما يكون (0) من القوة $\rightarrow$ بيجب

Not independent $\rightarrow$ إذا

$$W[x, x^2](0) = 0 \quad 0 \in (-\infty, \infty)$$

Is $\{0, 2x, 4x \sin x\}$ independent on (b, π) ?

No this set is dependent

إذا كان يوجد من في
الجزء لا تكون مستقلة

$$0 \in \{0, 2x, 4x \sin x\}$$

Find the value of k such that?

$\{2x, 2k-6, e^x\}$ is dependent

$$2k-6=0$$

$$k=3$$

$k=11-3=8$ independent

$$\begin{vmatrix} 2x & x \\ x & 1 \end{vmatrix} = 2x - x^2 = x(2-x)$$

$$(2-x) \neq 0 \quad \text{for } x \in (0, 2)$$

Fundamental set of solution

$\{y_1, y_2, \dots, y_n\}$ is said to be Fundamental set of solution for DE

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = 0$$

If ① $y_1, y_2, \dots, y_n$ are solution for DE

② $W[y_1, y_2, \dots, y_n] \neq 0$ (independent)

Abel's theorem

If y_1, y_2 are solution for

$$y'' + p(x) y' + q(x) y = 0 \quad \text{then}$$

$$W[y_1, y_2] = C e^{-\int p(x) dx}$$

Ex: If y_1, y_2 are solution for $xy'' + 2y' + xy = 0$

Find $W[y_1, y_2](x)$

$$y'' + \frac{2}{x} y' + y = 0$$

$$W[y_1, y_2](x) = C e^{-\int \frac{2}{x} dx}$$

$$= C e^{-2 \ln x} = \frac{C}{x^2}$$

Ex: If $x^2 y'' - 2y' + (3+x)y = 0$ if

1. $w[y_1, y_2](2) = 3$ Find: $w[y_1, y_2](5) = ?$

$$y'' - \frac{2}{x^2} y' + \frac{(3+x)}{x^2} y = 0$$

$$w[y_1, y_2](x) = C e^{-\int p(x) dx} = C e^{-\frac{2}{x}}$$

$$w[y_1, y_2](2) = C e^{-\frac{1}{2}} = 3 \Rightarrow C = 3e^{\frac{1}{2}}$$

$$w[y_1, y_2](5) = 3e^{\frac{1}{2}} e^{-\frac{2}{5}} = 3e^{\frac{3}{5}}$$

Reduction of order: If y_1 is a solution for DE

if y_1 is the solution for DE

$y'' + p(x)y' + q(x)y = 0$, then the other solution is

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$$

and the general solution is $y = C_1 y_1 + C_2 y_2$
Constant

Ex 8 If $y_1 = e^x$ is a solution for DE

$$xy'' + (x+1)y' + y = 0 \quad x > 0$$

Find the second solution and the general solution

$$y'' + \frac{(x+1)}{x} y' + \frac{y}{x} = 0$$

$$y_2 = e^x \cdot \int \frac{e^{-\int \frac{x+1}{x} dx}}{e^{2x}} dx = e^x \int \frac{e^{x+2\ln x}}{e^{2x}} dx$$

$$= e^x \int x e^x dx = e^x [-x e^{-x} - e^{-x}] = -x - 1$$

$$\text{general solution} = y = C_1 y_1 + C_2 y_2$$

$$C_1 e^x + C_2 (-x - 1)$$

Ex 8 If $y_1 = \cos(\ln x)$ is a solution for DE

$x^2 y'' + xy' + y = 0$ Find the general solution

9. $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$

$$y_2 = \cos(\ln x) \cdot \int \frac{e^{-\int \frac{1}{x} dx}}{\cos^2(\ln x)} dx = \cos(\ln x) \cdot \int \frac{1}{\cos^2(\ln x)} dx$$

$$= \cos(\ln x) \int \frac{1}{x \cos^2(\ln x)} dx = \cos(\ln x) \cdot \tan(\ln x)$$

$$= \sin \ln x \Rightarrow y = C_1 \cos(\ln x) + C_2 \sin(\ln x)$$

homogeneous, linear, second order with constant coefficients

$$ay'' + by' + cy = 0 \quad a \neq 0$$

constants

To solve this DE we suppose that the solution has the form $y = e^{rx}$

$$y' = re^{rx} \text{ and } y'' = re^{rx} \text{ put this in DE}$$

$$ay'' + by' + cy = 0$$

$$ar^2 e^{rx} + br e^{rx} + c e^{rx} = 0$$

$$(ar^2 + br + c) e^{rx} = 0$$

معنى
معنى

المعادلة
المصاحبة
the auxiliary
equation for DE

نم فصل المعادلة التفاضلية ونجد الجذور

then there will be 3 forms of the general solution

⊗ If $\Delta = b^2 - 4ac > 0$ then we have 2 distinct real roots r_1, r_2

then the solutions of DE are

$$y_1 = e^{r_1 x} \quad y_2 = e^{r_2 x}$$

general solution is $y = C_1 y_1 + C_2 y_2$

⊗

$$y'' + 5y' + 6y = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r = -3$$

$$r = -2$$

$$y_1 = e^{-2x}$$

$$y_2 = e^{-3x}$$

the general solution = $C_1 e^{-2x} + C_2 e^{-3x}$

$$\text{Solve } y'' - 4y = 0$$

$$r^2 - 4 = 0$$

$$\boxed{r = \pm 2}$$

$$y_1 = e^{-2x} \quad y_2 = e^{2x}$$

$$y = C_1 e^{-2x} + C_2 e^{2x}$$

$$\text{Solve } y'' - 5y' = 0$$

$$r^2 - 5r = 0$$

$$r = 0$$

$$r = 5$$

$$y_1 = e^0 = 1$$

$$y_2 = e^{5x}$$

$$\boxed{y = C_1 + C_2 e^{5x}}$$

⊗ $\Delta = b^2 - 4ac = 0$ - then we have one repeated

real root, say r_1, r_1

$$\Rightarrow y_1 = e^{r_1 x}$$

$$y_2 = x e^{r_1 x}$$

$$y = C_1 y_1 + C_2 y_2$$

$$\text{Solve } y'' + 12y' + 36y = 0$$

$$r^2 + 12r + 36 = 0$$

$$(r+6)(r+6) = 0$$

$$\boxed{r = -6}$$

$$y_1 = e^{-6x}$$

$$y_2 = x e^{-6x}$$

$$y = C_1 e^{-6x} + C_2 x e^{-6x}$$

$$4y'' + 20y' + 25y = 0$$

$$4r^2 + 20r + 25 = 0$$

$$(2r + 5)(2r + 5) = 0$$

$$\boxed{r = -\frac{5}{2}} \quad y_1 = e^{-\frac{5}{2}x} \quad y_2 = x e^{-\frac{5}{2}x}$$

$$y = c_1 e^{-\frac{5}{2}x} + c_2 x e^{-\frac{5}{2}x}$$

⊗ If $\Delta = b^2 - 4ac < 0$ then we have

complex roots say $r = \alpha \pm \beta i$

(where $\alpha, \beta \in \mathbb{R}$, $i = \sqrt{-1}$)

$$y_1 = e^{\alpha x} \cos(\beta x) \quad y_2 = e^{\alpha x} \sin(\beta x)$$

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$

$$\text{Solve } y'' + 2y' + 5y = 0$$

$$r^2 + 2r + 5 = 0$$

$$a = 1 \quad b = 2 \quad c = 5$$

$$\Delta = b^2 - 4ac = 2^2 - 4 \times 1 \times 5 = -16 < 0$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm \sqrt{-16}}{2 \times 1}$$

complex

$$= \frac{-2 \pm 4i}{2} = \boxed{-1 \pm 2i}$$

$$\sqrt{-1} = i$$

$$\sqrt{-4} = 2i$$

$$\sqrt{-9} = 3i$$

$$y_1 = e^{ax} \cos(Bx) = e^{-x} \cos(2x)$$

$$y_2 = e^{ax} \sin(Bx) = e^{-x} \sin(2x)$$

$$y = C_1 e^{-x} \cos(2x) + C_2 e^{-x} \sin(2x)$$

Solve: $y'' + 4y' + 7y = 0$

$$r^2 + 4r + 7 = 0$$

$$a=1, B=4, c=7$$

$$\Delta = (4)^2 - 4 \cdot 7 = -12 < 0$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2\sqrt{3}i}{2}$$

$$= -2 \pm \sqrt{3}i \Rightarrow y_1 = e^{-2x} \cos(\sqrt{3}x) \quad y_2 = e^{-2x} \sin(\sqrt{3}x)$$

$$y = C_1 e^{-2x} \cos(\sqrt{3}x) + C_2 e^{-2x} \sin(\sqrt{3}x)$$

Solve the following DEs

$$\textcircled{1} y'' + 5y' + 6y = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+1)(r+6) = 0$$

$$r_1 = -1$$

$$r_2 = -6$$

$$\Rightarrow y_1 = e^{-x}$$

$$y_2 = e^{-6x}$$

$$y = C_1 e^{-x} + C_2 e^{-6x}$$

$$\textcircled{2} y'' - 10y' + 25y = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r-5)(r-5) = 0 \quad \boxed{r=5}$$

$$y_1 = e^{5x}$$

$$y_2 = x e^{5x}$$

$$\Rightarrow y = C_1 e^{5x} + C_2 x e^{5x}$$

$$\textcircled{3} y'' - 2y' + 5y = 0$$

$$r^2 - 2r + 5 = 0$$

$$\Delta = 4 - 4 \pm 5 \times 1 = -16 < 0$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$y_1 = e^x \cos(\sqrt{3}x) / y_2 = e^x \sin(\sqrt{3}x) \Rightarrow y = C_1 e^x \cos(\sqrt{3}x) + C_2 e^x \sin(\sqrt{3}x)$$

$$y'' + 9y = 0$$

$$r^2 + 9r = 0$$

$$\boxed{r = \pm 3i}$$

$$y_1 = e^{0} \cos 3x = \cos 3x$$

$$y_2 = \sin 3x$$

$$y = C_1 \cos(3x) + C_2 \sin(3x)$$

$$y'' + 6y' + 13y = 0$$

$$r^2 + 6r + 13 = 0$$

$$\Delta = 36 - 4 \times 13 = -16 < 0 \Rightarrow r = \frac{-6 \pm \sqrt{-16}}{2}$$

$$\frac{-6 \pm 4i}{2} \Rightarrow y_1 = e^{18x} \cos(2x) \quad y_2 = e^{18x} \sin(2x)$$

$$y'' + 4y' = 0$$

$$r^2 + 4r = 0$$

$$r = 0 \quad r = -4$$

$$y_1 = e^{0x} = 1$$

$$y_2 = e^{-4x}$$

$$y = C_1 + C_2 e^{-4x}$$

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0 \quad r = -1$$

$$y_1 = e^{-1x}$$

$$y_2 = xe^{-1x}$$

$$y = C_1 e^{-1x} + C_2 x e^{-1x}$$

If $y_1 = e^{-5x}$ $y_2 = e^x$ are solutions for DE

$$y'' + (b-3)y' - 4cy = 0 \quad \text{Find } b, c$$

$$r_1 = -5 \quad r_2 = 1$$

$$(r+5)(r-1) = 0$$

$$r^2 - r + 5r - 5 = 0 \Rightarrow r^2 + 4r - 5 = 0$$

$$b-3 = 4$$

$$-4c = -5$$

$$b = 7$$

$$c = \frac{5}{4}$$

$$y = C_1 e^{-5x} + C_2 e^x$$

و مكن يحكيه

Find the DE $ay'' + by' + cy = 0$ if
has $y = xe^{3x}$ a solution

$$y_1 = e^{3x}$$

$$y_2 = xe^{3x}$$

$$\boxed{r = 3}$$

$$(r-3)^2 = 0$$

$$r^2 - 6r + 9 = 0$$

$$a = 1 \quad b = -6 \quad c = 9$$

If $y = e^{-3x} \sin(2x)$ is a solution for DE
 $ay'' + (2b-3)y' - 2y = 0$ Find a, b, c

$$\alpha = -3$$

$$\beta = 2$$

$$\boxed{r = -3 \pm 2i}$$

$$r + 3 = \pm 2i$$

$$(r+3)^2 = (\pm 2i)^2$$

$$r^2 + 6r + 9 = 4i^2 \Rightarrow r^2 + 6r + 9 = -4$$

$$\boxed{r^2 + 6r + 13 = 0}$$

$$\boxed{a = 1}$$

$$2b - 3 = 6$$

$$\boxed{c = -13}$$

$$\boxed{b = \frac{9}{2}}$$

Solve the following IVPs

$$y'' - 3y' + 2y = 0$$

$$y(0) = 3$$

$$y'(0) = -5$$

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0$$

$$r=1$$

$$r=2$$

$$\Rightarrow y_1 = e^x$$

$$y_2 = e^{2x}$$

$$y = C_1 e^x + C_2 e^{2x}$$

$$y(0) = C_1 + C_2 = 3$$

$$y' = C_1 e^x + 2C_2 e^{2x}$$

$$y'(0) = (C_1 + 2C_2 = -5) \quad \times 2$$

$$-2C_2 = 8$$

$$C_2 = -8$$

$$C_1 = 11$$

$$C_1 = 11$$

Solution is $y = -8e^{2x} + 11e^x$

How to solve the following IVPs

$$y'' + 2y' + 5y = 0$$

$$y(0) = 3$$

$$y'(0) = -2$$

Linear homogeneous higher order DE with constant coefficient

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \dots, a_n$ are constants

Ex: solve the following DEs

1) $y''' - 2y'' - 3y' = 0$

$$r^3 - 2r^2 - 3r = 0$$

$$r(r^2 - 2r - 3) = 0$$

$$r = 0 \quad r = +3 \quad r = -1$$

$$y_1 = e^{0x} = 1 \quad y_2 = e^{3x} \quad y_3 = e^{-x}$$

$$y = C_1 + C_2 e^{3x} + C_3 e^{-x}$$

2) $y''' + 3y'' + 3y' + y = 0$

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$(r^3 + 1) + 3r(r+1) = 0$$

$$(r+1)(r^2 - r + 1) + 3r(r+1) = 0$$

$$(r+1)(r^2 - r + 1 + 3r) = 0$$

$$(r+1)(r^2 + 2r + 1) = 0$$

$$r_1 = -1 \quad r_2 = -1 \quad r_3 = -1$$

$$y_1 = e^{-x} \quad y_2 = x e^{-x} \quad y_3 = x^2 e^{-x}$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} \quad \text{--- "y d -" y ①}$$

$$\boxed{3} \quad y^{(3)} - 8y = 0 \quad \text{--- } \lambda = 0, 1, 1 + \sqrt{-3}, 1 - \sqrt{-3} \quad \text{--- } \boxed{1-2}$$

$$r^3 - 8 = 0 \rightarrow (r-2)(r^2 + 2r + 4) = 0$$

$$r = 2 \quad \left| \quad r = \frac{4 \pm \sqrt{4 - 4 \times 1 \times 4}}{2} = \frac{4 \pm \sqrt{-12}}{2}$$

$$2 \pm \sqrt{3}i$$

$$y_1 = e^{2x} \quad y_2 = e^{2x} \cos(\sqrt{3}x) \quad y_3 = e^{2x} \sin(\sqrt{3}x)$$

$$y = C_1 e^{2x} + C_2 e^{2x} \cos(\sqrt{3}x) + C_3 e^{2x} \sin(\sqrt{3}x)$$

$$\boxed{4} \quad y^{(3)} + 27y = 0$$

$$r^3 + 27 = 0$$

$$(r+3)(r^2 - 3r + 9) = 0$$

$$r = -3 \quad \left| \quad r = \frac{3 \pm \sqrt{9 - 4 \times 9}}{2} = \frac{3 \pm \sqrt{-27}}{2}$$

$$\frac{3 \pm 3\sqrt{3}i}{2}$$

$$\alpha = \frac{3}{2} \quad \beta = 3\sqrt{3}$$

$$y_1 = e^{-3x} \quad y_2 = e^{\frac{3}{2}x} \cos(3\sqrt{3}x) \quad y_3 = e^{\frac{3}{2}x} \sin(3\sqrt{3}x)$$

Ex 8 Solve the following DE:

① $y''' - 6y'' + 11y' - 6y = 0$

$$r^3 - 6r^2 + 11r - 6 = 0$$

$r=1 \rightarrow 1^3 - 6(1)^2 + 11(1) - 6 = 0 \checkmark$

r^3	r^2	r	
1	-6	11	-6
1	1	-5	17
		6	-46
			0

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$r=3$ $r=2$

$y_1 = e^x$ $y_2 = e^{3x}$ $y_3 = e^{2x}$

$$y = C_1 e^x + C_2 e^{3x} + C_3 e^{2x}$$

② $y^{(3)} + 3y^{(2)} + 4y = 0$

$r=1$

$$r^3 + 3r^2 + 4r = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

r^3	r^2	r	
1	3	4	0
1	4	4	4
		0	0

$y_1 = e^x$

$y_2 = e^{-2x}$

$y_3 = x e^{-2x}$

$$y = C_1 e^x + C_2 e^{-2x} + C_3 x e^{-2x}$$

$$(3) \quad y^{(5)} - y^{(4)} - 2y^{(3)} = 0$$

$$r^5 - r^4 - 2r^3 = 0$$

$$r = -1$$

$$\begin{array}{cccccc} r^5 & r^4 & r^3 & r^2 & r & 1 \\ 1 & -1 & -2 & 0 & 0 & 0 \\ & 1 & -1 & 2 & 0 & 0 \\ & & 1 & -2 & 0 & 0 \\ & & & 1 & -2 & 0 \\ & & & & 1 & -2 \\ & & & & & 1 \end{array}$$

$$r^4 - 2r^3 = 0$$

$$r^3(r - 2) = 0$$

$$r_1 = 0$$

$$r_4 = 2$$

$$y_1 = 1$$

$$y_2 = x$$

$$y_3 = x^2$$

$$r_2 = 0$$

$$y_4 = e^{2x}$$

$$y_5 = e^{-x}$$

$$r_3 = 0$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{2x} + C_5 e^{-x}$$

$$(4) \quad y^{(6)} - 8y^{(4)} + 16y^{(2)} = 0$$

$$r^6 - 8r^4 + 16r^2 = 0$$

$$r^2(r^4 - 8r^2 + 16) = 0$$

$$r^2(r^2 - 4)(r^2 - 4) = 0$$

$$\downarrow$$

$$\downarrow$$

$$\downarrow$$

$$r_1 = 0$$

$$r_3 = 2$$

$$r_5 = 2$$

$$r_2 = 0$$

$$r_4 = -2$$

$$r_6 = -2$$

$$y_1 = e^0 = 1$$

$$y_2 = x$$

$$y_3 = e^2$$

$$y_4 = e^{-2}$$

$$y_5 = xe^2$$

$$y_6 = xe^{-2}$$

$$y = C_1 + C_2 x + C_3 e^2 + C_4 e^{-2} + C_5 xe^2 + C_6 xe^{-2}$$

$$(5) \quad y^{(4)} - 81y = 0$$

$$r^4 - 81 = 0$$

$$(r^2 - 9)(r^2 + 9) = 0$$

$$\downarrow$$

$$r_1 = 3$$

$$r_2 = -3$$

$$r = \pm \sqrt{-9} = \pm 3i$$

$$\alpha = 0, \quad \beta = 3$$

$$y_3 = e^0 \cos(3x) \quad y_4 = e^0 \sin(3x)$$

$$y_1 = e^{3x}$$

$$y_2 = e^{-3x}$$

$$y_3 = \cos(3x)$$

$$y_4 = \sin(3x)$$

$$y = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos(3x) + C_4 \sin(3x)$$

$$(6) \quad y^{(4)} = 0$$

$$r^4 = 0$$

$$r_1 = 0$$

$$r_2 = 0$$

$$r_3 = 0$$

$$r_4 = 0$$

$$y_1 = 1$$

$$y_2 = x$$

$$y_3 = x^2$$

$$y_4 = x^3$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3$$

$$(7) \quad y^{(4)} + 4y'' + 4y = 0$$

If $y = C_1 e^{2x} + C_2 e^{-5x} + C_3 x e^{-5x}$ is a general

solution for a linear homogen DE with constant coefficients

Find this DEs

$$r_1 = 2 \quad r_2 = -5 \quad r_3 = -5$$

$$(r-2)(r+5)(r+5) = 0$$

$$(r-2)(r+5)^2 = 0$$

$$(r-2)(r^2 + 10r + 25) = 0$$

$$r^3 + 10r^2 + 25r - 2r^2 - 20r - 50 = 0$$

$$r^3 + 8r^2 + 5r - 50 = 0$$

$$y''' + 8y'' + 5y' - 50y = 0$$

Given $(r+4)(r^2-1)^2(r^2+4)^2 = 0$ is the auxiliary equation for linear, homog. DE with constant coefficients

① Find the order of this DE?

② Find the general solution of this DE? ?

$$r+4 = 0$$

$$\boxed{r = -4}$$

$$(r^2-1)^2 = 0$$

$$\boxed{r = \pm 1}$$

so

$$(r^2+4)^2 = 0$$

$$r = \pm 2i$$

so

$$y_1 = e^{-4x}$$

$$y_2 = x$$

$$y_3 = x^{-1}$$

$$y_4 = e^x$$

$$y_5 = x e^{-x}$$

$$y_6 = \cos(2x)$$

$$y_7 = \sin(2x)$$

$$y_8 = x \cos(2x)$$

$$y_9 = x \sin(2x)$$

Cauchy-Euler Equation (homogenous)

$$ax^2y'' + bxy' + cy = 0 \quad x > 0$$

ليست ثابت

then the substitution $x = e^t \Leftrightarrow \ln x = t$ transform the Cauchy-Euler eq. into $a \frac{d^2y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$

Exs solve the following DEs

$$\square x^2y'' - 2xy' - 4y = 0 \quad x > 0$$

$$a = 1$$

$$x = e^t$$

$$b = -2$$

$$\ln x = t$$

$$c = -4$$

$$\frac{d^2y}{dt^2} + -3 \frac{dy}{dt} - 4y = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$r = 4 \quad r = -1$$

$$y_1(t) = e^{4t} \rightarrow x^4$$

$$y_2(t) = e^{-t} \rightarrow x^{-1}$$

$$y = c_1 x^4 + c_2 x^{-1}$$

Ex 8 $(2xy'' - y' - \frac{2}{x}y = 0) \quad x > 0$

$$2x^2 y'' - xy' - 2y = 0 \quad \text{Euler eq}$$

$x = e^t \leftrightarrow \ln x = t \quad a = 2 \quad b = -1 \quad c = -2$

$$\frac{a d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0 \quad c = -2$$

$$2 \frac{d^2 y}{dt^2} + -3 \frac{dy}{dt} - 2y = 0$$

$$2r^2 - 3r - 2 = 0$$

$$(2r+1)(r-2) = 0$$

$r_1 = -\frac{1}{2} \quad r_2 = 2 \Rightarrow y_1(t) = e^{-\frac{1}{2}t} \quad y_2(t) = e^{2t}$

$y_1(x) = x^{-\frac{1}{2}} \quad y_2(x) = x^2$
 $y = C_1 x^{-\frac{1}{2}} + C_2 x^2$

$(x^3 y'' + x^2 y' + 4xy = 0) \div x \quad x > 0$

$$x^2 y'' + xy' + 4y = 0 \Rightarrow \text{Euler Eq}$$

$x = e^t \leftrightarrow \ln x = t \quad a = 1 \quad b = 1 \quad c = 4$

$$ar^2 + (b-a)r + c = 0$$

$$r^2 + 0 + 4 = 0 \Rightarrow r = \pm \sqrt{-4} = \pm 2i$$

$$\alpha = 0$$

$$\beta = 2$$

$$y_1(t) = e^0 \cos(2t)$$

$$y_2(t) = e^0 \sin(2t)$$

$$y_1(x) = \cos(2 \ln x)$$

$$y_2(x) = \sin(2 \ln x)$$

$$y = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$$

$$x^2 y'' + 3xy' + 3y = 0 \Rightarrow \text{Euler Eq}$$

$$x = e^t \Leftrightarrow \ln x = t$$

$$a = 1$$

$$b = 3$$

$$c = 3$$

$$r^2 + 2r + 3 = 0 \Rightarrow \Delta = b^2 - 4ac = 4 - 4 \times 1 \times 3 = 4 - 12 = -8 < 0$$

$$4 - 12 = -8 < 0$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \frac{-2 \pm \sqrt{-8}}{2 \times 1} = \frac{-2 \pm 2\sqrt{2}i}{2}$$

(complex)

$$r_1 = -1 + \sqrt{2}i \rightarrow y_1(t) = e^{-t} \cos(\sqrt{2} t)$$

$$r_2 = -1 - \sqrt{2}i \rightarrow y_2(t) = e^{-t} \sin(\sqrt{2} t)$$

$$y_1(x) = x^{-1} \cos(\sqrt{2} \ln x)$$

$$y_2(x) = x^{-1} \sin(\sqrt{2} \ln x)$$

$$y = C_1 x^{-1} \cos(\sqrt{2} \ln x) + C_2 x^{-1} \sin(\sqrt{2} \ln x)$$

$$4x^2 y'' + 8xy' + y = 0 \rightarrow \text{Euler Eq}$$

$$x = e^t \leftrightarrow \ln x = t \quad a = 4 \quad b = 8 \quad c = 1$$

$$4r^2 + 4r + 1 = 0$$

$$(2r + 1)^2 = 0$$

$$\text{So } \boxed{r = -\frac{1}{2}}$$

$$y_1(t) = e^{\frac{-1}{2}t} \rightarrow y_1(x) = e^{\frac{-1}{2}\ln x} = x^{-\frac{1}{2}}$$

$$y_2(t) = t e^{\frac{-1}{2}t} \rightarrow y_2(x) = \ln x \cdot x^{-\frac{1}{2}}$$

$$y = c_1 e^{\frac{-1}{2}\ln x} + c_2 \ln x \cdot x^{-\frac{1}{2}}$$

Solution $\rightarrow$

Ex 8 If $y = x^3 \ln x$ is a solution for DE

$$ax^2 y'' + bxy' + cy = 0, \quad x > 0$$

Find a, b, c ?

$$x = e^t \leftrightarrow \ln x = t$$

$$y(t) = e^{3t} \cdot t = t e^{3t}$$

$$r_1 = 3$$

$$r_2 = 3$$

$$(r - 3)^2 = 0$$

$$r^2 - 6r + 9 = 0$$

$$y'' - 6y' + 9y = 0$$

$$\boxed{a = 1}$$

$$\boxed{c = 9}$$

$$b - a = 1 - 6$$

$$\boxed{b = -5}$$

If $y = x^2 \cos(2 \ln x)$ is a solution for DE $x^2 y'' + bxy' + cy = 0$

Find a, b, c ?

$$x = e^t \rightarrow \ln x = t$$

$$y = x^2 \cos(2 \ln x) = e^{2t} \cos(2t) \quad (\text{Complex})$$

$$r = \alpha + \beta i$$

$$r = 2 \pm 2i$$

$$r - 2 = \pm 2i$$

$$(r - 2)^2 = -4$$

$$r^2 - 4r + 4 = -4$$

$$r^2 - 4r + 8 = 0$$

$$y'' - 4y' + 8y = 0 \Rightarrow \boxed{a=1} \quad \boxed{b=-4} \quad \boxed{c=8}$$

$$\boxed{b=-3}$$

If $\omega[2, x, f(x)] = 8x$ Find the function $f(x)$ where

$$f(0) = 2 \quad f'(0) = 3$$

$$\omega[2, x, f(x)] = 8x$$

x	$f(x)$
0	2
1	3
0	0

$$= 8x \Rightarrow 2 \times 1 \times f''(x) = 8x$$

$$\boxed{f''(x) = 4x}$$

$$\int f''(x) = \int 4x dx \Rightarrow f'(x) = 2x^2 + c$$

$$f'(0) = \boxed{c=3}$$

$$\int f'(x) = \int 2x^2 + 3$$

$$\boxed{f(x) = \frac{2}{3}x^3 + 3x + C}$$

$$f(x) = \frac{2}{3}x^3 + 3x + C$$

$$f(0) = 2$$

$\Rightarrow$

$$\boxed{f(x) = \frac{2}{3}x^3 + 3x + 2}$$

If $\omega [x, y(x)] = x^2 e^x$ find the function $y(x)$ where $y(0) = e^2$

$$\omega [x, y(x)] = x^2 e^x$$

x	$y(x)$
1	$y'(x)$

$$= x^2 e^x$$

$$x y'(x) - y(x) = x^2 e^x$$

$$y'(x) - \frac{y(x)}{x} = x e^x \quad (\text{linear})$$

$$M(x) = e^{\int p(x) dx} = e^{\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$y = \frac{1}{M(x)} \int M(x) g(x) dx + \frac{C}{M(x)}$$

$$y = \frac{1}{\frac{1}{x}} \int \frac{1}{x} \cdot x e^x dx + \frac{C}{\frac{1}{x}}$$

$$y = x e^x + xC$$

$$y(1) = e + C = e^2$$

$$C = e^2 - e \Rightarrow y = x e^x + e^2 - e$$

Remark 8 The DE $ay'' + by' + cy = 0$ can be written as $L[y](x) = 0$ where $L[y](x) = ay'' + by' + cy$ is a Linear operator

Ex 8 If $L[y](x) = 3y'' - 4y' + 5y$ Find $L[y](x)$ where $y = x^3 - 4x^2 + 1$

$$\text{Sol: } y' = 3x^2 - 8x$$

$$y'' = 6x - 8$$

$$L[y](x) = L[x^3 - 4x^2 + 1]$$

$$= 3(6x - 8) - 4(3x^2 - 8x) + 5(x^3 - 4x^2 + 1)$$

$$= 18x - 24 - 12x^2 - 32x + 5x^3 - 20x^2 + 5$$

$$= 5x^3 - 32x^2 + 50x - 19$$

Chapter 4 :

Non homoge. Linear second and higher order

$$ay'' + by' + cy = g(x)$$

to solve this DE :

① Solve the homog. equation $ay'' + by' + cy = 0$
to find the homog. solution say : $y_h = c_1 y_1 + c_2 y_2$

② solve the non homoge to find the particular solution
say y_p

③ the general solution = $y_h + y_p$

To find the particular solution y_p , we use
the following methods :

① Undetermined coefficients

② Variation of parameter

Method ① : Undetermined coefficients

$$ay'' + by' + cy = g(x)$$

To use this method $g(x)$ must be polynomial/exponential/
or $\sin, \cos$.

$g(x) \rightarrow \text{polynomial}$

5 مرتبة $\rightarrow$ x^k مرتبة (A)

$x+2$ خطي $\rightarrow (Ax+B)$

x^2 تربيعي $\rightarrow (Ax^2+Bx+C)$

$k = \text{مرتبة جذور}$

المرتبة (جذور)

العادية

$g(x) \rightarrow \text{exponential}$

$e^{\alpha x} \rightarrow A e^{\alpha x} \cdot x^m$

$x e^{\alpha x} \rightarrow (Ax+B) e^{\alpha x} \cdot x^m$

$x^2 e^{\alpha x} \rightarrow (Ax^2+Bx+C) e^{\alpha x} \cdot x^m$

$m = \text{مرتبة جذور}$

المرتبة α في جذور العادية

$g(x) \rightarrow \sin / \cos$

$2 \sin(Bx) \rightarrow (A \cos(Bx) + B \sin(Bx)) x^n$

$x \sin(Bx) \rightarrow ((Ax+B) \cos(Bx) + (Cx+D) \sin(Bx)) x^n$

$x e^{\alpha x} \sin(Bx) \rightarrow ((Ax+B) e^{\alpha x} \cos(Bx) + (Cx+D) e^{\alpha x} \sin(Bx)) x^n$

$\alpha + Bi = r$ مرتبة جذور المعادلة

في جذور المعادلة

Find the general solution for the DE

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

وصيغة y

soln

① homoge.

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = 3 \quad r = -1$$

$$y_1 = e^{3x}$$

$$y_2 = e^{-x}$$

$$y_h = C_1 e^{3x} + C_2 e^{-x}$$

② non homog.

$$g(x) = 4x - 5 + 6xe^{2x}$$

$$y_p = (Ax + B) X^k + (Cx + D) e^{2x} X^m$$

دالة X^k

مشتقات

فرد X^k

فرد X^m

فرد X^2

منفرد = (1) فرد

$$y_p = (Ax + B) + (Cx + D)e^{2x} \rightarrow \text{The form of particular solution}$$

substituting to find A, B, C, D

1st

$$y_p' = A + 2(Cx + D)e^{2x}$$

$$y_p'' = 2Ce^{2x} + 2(Cx + D)e^{2x} + 4(Cx + D)e^{2x}$$

نحوه من في المعادلة لا ملية

$$y'' + 2y' - 3y = 4x - 5 + 6xe^{2x}$$

$$4C e^{2x} + 4(Cx + D)e^{2x} - 2(A + Ce^{2x} + 2(Cx + D)e^{2x}) =$$

$$3(Ax + B + (Cx + D)e^{2x}) = 4x - 5 + 6xe^{2x}$$

$$= \underbrace{-2A - 3B}_{\text{constant}} + \underbrace{(2C - 3D)e^{2x}}_{e^{2x}} - \underbrace{3Ax}_{x} - \underbrace{3Cxe^{2x}}_{xe^{2x}} = \underbrace{4x}_{x} - \underbrace{5}_{\text{constant}} + \underbrace{6xe^{2x}}_{xe^{2x}}$$

$$-2A - 3B = -5 \rightarrow \boxed{B = \frac{7}{9}}$$

$$2C - 3D = 0 \rightarrow \boxed{D = \frac{-4}{3}}$$

$$-3A = 4 \rightarrow \boxed{A = \frac{-4}{3}}$$

$$-3C = 6 \rightarrow \boxed{C = -2}$$

the particular sol is

$$y_p = \frac{-4}{3}x + \frac{7}{9} + \left(-2x - \frac{4}{3}\right)e^{2x}$$

the general solution is

$$y = \underbrace{\frac{-4}{3}x + \frac{7}{9} + \left(-2x - \frac{4}{3}\right)e^{2x}}_{y_p} + \underbrace{C_1 e^{3x} + C_2 e^{-x}}_{y_h}$$

Exo Find the general solution for DE

$$y'' + 4y = \cos(2x)$$

① homog : $r^2 + 4 = 0$ $r = \pm 2i$ $\begin{cases} y_1 = \cos(2x) \\ y_2 = \sin(2x) \end{cases}$

$y_h = C_1 \cos(2x) + C_2 \sin(2x)$

② non homog : $q(x) = \cos(2x)$

$y_p = (A \cos(2x) + B \sin(2x)) \times$ x (n) $\rightarrow$ $r = \alpha + \beta i$
 $10 + 2i$

فرضت حصة واحدة في (r)

$y_p = A x \cos(2x) + B x \sin(2x)$

derivative
particular sol

to find A, B

$$y'_p = A \cos(2x) - 2Ax \sin(2x) + B \sin(2x) + 2Bx \cos(2x)$$

$$y''_p = -2A \sin(2x) - 2A \sin(2x) - 4Ax \cos(2x) + 2B \cos(2x) + 2B \cos(2x) - 4Bx \sin(2x)$$

نقول

$$-4A \sin(2x) + 4B \cos(2x) = \cos(2x)$$

sin jobs

$$-4A = 0$$

$A = 0$

cos jobs

$$4B = 1$$

$B = \frac{1}{4}$

~~$y_p = \cos(2x)$~~ $y_p = \frac{x \sin(2x)}{4}$

$$y = C_1 \cos(2x) + C_2 \sin(2x) + \frac{x \sin(2x)}{4}$$

Find a form of particular solution for the following DE without calculating the constant:

$$\textcircled{1} y'' + -5y' + 6y = 2 \sin 4x + 5e^{2x}$$

homog: $y'' - 5y' + 6y = 0$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r=2$$

$$r=3$$

$$y_1 = e^{2x}$$

$$y_2 = e^{3x}$$

non-homog: $g(x) = 2 \sin 4x + 5e^{2x}$

$$y_p = (A \cos 4x + B \sin 4x) X^n + (C e^{2x}) X^m$$

$\alpha = (\alpha + \beta i)$
 $4i$

فرض المص

درجات
م
درجات
م
درجات
م

$$y_p = A \cos 4x + B \sin 4x + X C e^{2x}$$

$$\textcircled{2} y'' + 2y' = e^x - y$$

$$y'' + 3y = e^x$$

$$y + y'' + 2y' = 0$$

$$\Rightarrow r + r^2 + 2r = 0 \Rightarrow r = -1, r = -1$$

$$y_1 = e^{-x} \quad y_2 = -e^{-x}$$

$$y_p = (A e^{-x}) X^m$$

$$m = 2$$

$$y_p = Ax^2 e^{-x}$$

$$(3) y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$$r_1 = 3$$

$$r_2 = 3$$

$$y_1 = e^{3x}$$

$$y_2 = xe^{3x}$$

$$g(x) = 6x^2 + 2 - 12e^{3x}$$

$$y_p = (Ax^2 + Bx + C)x^{(k)} + (De^{3x})x^{(m)}$$

0 = 10 ← حالات تكرر
2 = ← حالات تكرر

$$y_p = Ax^2 + Bx + C + Dx^2 x^{3x}$$

$$(4) y^{(4)} - 2y^{(3)} + y^{(2)} = 1 + e^x$$

$$r^4 - 2r^3 + r^2 = 0$$

$$r^2(r-2r+1) = 0$$

$$r=0$$

$$r=0$$

$$r=1$$

$$r=1$$

$$y_1 = 1$$

$$y_2 = x$$

$$y_3 = e^x$$

$$y_4 = xe^x$$

$$g(x) = 1 + e^x$$

$$y_p = (Ax^2)x^{(n)} + (Be^x)x^{(k)}$$

$$y_p = Ax^2 + Bx^2 e^x$$

$$⑤ \quad y''' + 4y'' + 3y' = x^2 \cos x - 3x e^{-3x}$$

$$r^3 + 4r^2 + 3r = 0$$

$$r(r^2 + 4r + 3) = 0$$

$$r = 0 \quad r = -3 \quad r = -1$$

$$y_1 = 1 \quad y_2 = x^{-3} \quad y_3 = e^{-1x}$$

$$g(x) = \underbrace{x^2 \cos x}_{(n)} - \underbrace{3x e^{-3x}}_{(m)}$$

$$0 = \alpha + \beta i$$

$$y_p = (Ax^2 + Bx + C) \cos x + (Dx^2 + Ex + G) \sin x + (Wx^2 + Vx) e^{-3x}$$

$$(Ax^2 + Bx + C) \cos x + (Dx^2 + Ex + G) \sin x + (Wx^2 + Vx) e^{-3x}$$

$$y^{(3)} - 2y^{(2)} + y^{(1)} = x + 2e^x - \cos(3x)$$

$$r^3 - 2r^2 + r = 0$$

$$r(r^2 - 2r + 1) = 0$$

$$\boxed{r=0} \quad \boxed{r=1} \quad \boxed{r=1}$$

$$g(x) = \underbrace{x}_{(1)} + \underbrace{2e^x}_{(2)} - \underbrace{\cos(3x)}_{(n)}$$

$$y_p = (Ax + B) x^{(1)} + (C e^x) x^{(2)} + (D \cos 3x + E \sin 3x) x^{(n)}$$

$$(Ax + B) x + C x^2 e^x + D \cos 3x + E \sin 3x$$

$$\textcircled{7} y'' + 4y = \cos^2 x$$

$$r^2 + 4r = \cos^2 x$$

$$r^2 = -4 \quad r = \pm 2i$$

$$y_1 = \cos 2x$$

$$y_2 = \sin 2x$$

$$g(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$y_p = A x^{(n)} + (B \cos 2x + C \sin 2x) x^{(n)}$$

$$y_p = A + B x \cos 2x + C x \sin 2x$$

$$\textcircled{8} y'' + y = \sin x \cos 2x$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$y_1 = \sin x \quad y_2 = \cos x$$

$$g(x) = \sin x \cos 2x = \frac{1}{2} [\sin(-x) + \sin(3x)]$$

$$= \frac{-1}{2} \sin x + \frac{1}{2} \sin 3x$$

$$y_p = (A \cos x + B \sin x) x^{(n_1)} + (D \cos 3x + C \sin 3x) x^{(n_2)}$$

$$y_p = A x \cos x + B x \sin x + D \cos(3x) + C \sin(3x)$$

$$\textcircled{9} \quad x^2 y'' + 3xy' - 8y = (\ln x)^3 - \ln x \quad (5)$$

Euler eq. $\Rightarrow x = e^t \leftrightarrow \ln x = t$

$$a = 1$$

$$b = 3$$

$$c = -8$$

$$a \frac{dy^2}{dt^2} + (b-a) \frac{dy}{dt} + cy = g(t)$$

$$\frac{dy^2}{dt^2} + 2 \frac{dy}{dt} - 8y = t^3 - t$$

homog

$$r^2 + 2r - 8 = 0$$

$$r = -4 \quad r = 2$$

$$y_1 = e^{-4t}$$

$$y_2(t) = e^{2t}$$

non

$$g(t) = t^3 - t$$

$$y_p(t) = (At^3 + bt^2 + ct + D) \quad \textcircled{k}$$

$$y_p(t) = At^3 + bt^2 + ct + D$$

$$A = -\frac{1}{8}$$

$$B = -\frac{3}{32}$$

$$y_p(x) = A \ln^3 x + b \ln^2 x + c \ln x + D$$

$$C = -\frac{1}{64}$$

$$D = -\frac{7}{250}$$

$$x^2 y'' - x y' + y = 2x$$

Euler eq

$$x = e^t \leftrightarrow \ln x = t$$

$$a=1 \quad b=-1 \quad c=1$$

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = g(t)$$

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} + y = 2e^t$$

$$\textcircled{*} \text{homog. eq} \quad r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0 \quad y_1 = e^t \quad y_2 = te^t$$

$$\textcircled{*} \text{non-homog. eq} \quad g(t) = 2e^t$$

$$y_p(t) = (A e^t) t^m$$

$$y_p(t) = a t^2 e^t \rightarrow y_p(x) = a x (\ln x)^2 \checkmark$$

$$\textcircled{11} x^2 y'' - x y' + y = \ln x + \sin(\ln x)$$

$$x = e^t \rightarrow \ln x = t$$

$$a=1 \quad b=1 \quad c=1$$

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = g(t)$$

$$\frac{d^2 y}{dt^2} + y = t + \sin t$$

$$\textcircled{*} r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_1 = \cos t \quad y_2 = \sin t$$

$$y_p(t) = (A t + B) t^{\text{م}} + (D \cos t + C \sin t) t^{\text{م}}$$

$$y_p(x) = A \ln x + B + D \ln x - C_1 (\ln x) + C_2 \ln x \sin(\ln x)$$

Find the consty. Euler eq of second order having general solution:

$$y(x) = C_1 x + C_2 x \ln x + x(\ln x)^2$$

Euler eqs $x = e^t$ $\ln x = t$

$$y(x) = \underbrace{C_1 e^t + C_2 t e^t}_{y_h} + \underbrace{e^t t^2}_{y_p}$$

homoge. $y_h = C_1 e^t + C_2 t e^t$

$$\Rightarrow r_1 = 1 \quad r_2 = 1$$

$$(r-1)^2 = 0$$

$$r^2 - 2r + 1 = 0$$

$$\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + y = 0 \Rightarrow a \frac{dy^2}{dt^2} + (b-a) \frac{dy}{dt}$$

$$a = 1$$

$$b = -1$$

$$c = 1$$

non-homog. $y_p = e^t t^2$

particular soln for:

$$\frac{dy^2}{dt^2} - 2 \frac{dy}{dt} + y = 9t$$

$$(y')_p = 2te^t + t^2e^t$$

$$(y'')_p = 2e^t + 2te^t + 2te^t + t^2e^t$$

نقوم

$$2e^t + 4te^t + t^2e^t$$

$$2e^t + 4e^t/e^t + t^2/e^t - 2(2te^t + t^2/e^t) + t^2e^t = g(t)$$

$$2e^t = g(t)$$

$$g(t) = 2x$$

$$\text{Euler eq} = ax^2y'' + bxy' + cy = g(x)$$

$$x^2y'' + xy' + y = g(x) \quad \checkmark$$

H.W 8 Solve the I.V.P

$$y''' + 2y'' + 9y' - 18y = -18x^2 - 18x + 22$$

$$y(0) = -2 \quad y'(0) = -8 \quad y''(0) = -12$$

① بحل Homog بـ a, b, c

method ② : Variation of parameters

$$ay'' + by' + cy = f(x)$$

① Find y_1, y_2 for homog D.E $ay'' + by' + cy = 0$

$$y_h = c_1 y_1 + c_2 y_2$$

② Find $w[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

③ Find $v_1 = - \int \frac{f(x) y_2}{w} dx$

$v_2 = \int \frac{f(x) y_1}{w} dx$

$$f(x) = \frac{f(x)}{a}$$

④ The particular solution is

$$y_p = y_1 v_1 + y_2 v_2$$

⑤ The general solution = $y_{\text{homog}} + y_{\text{particular}}$

Ex 8 Find the g.s for $y'' - 2y' + y = \frac{e^x}{x^2 + 1}$

① homog $r^2 - 2r + 1 = 0$ $r=1$ $r=1$ $y_1 = e^x$ $y_2 = xe^x$

$$y = c_1 e^x + c_2 x e^x$$

② $w[y_1, y_2] = w[e^x, x e^x] = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^{2x} + x e^{2x} - x e^{2x} = e^{2x}$

$$g(x) = \frac{e^x}{x^2+1} = \frac{e^x}{x^2+1}$$

$$v_1' = - \int \frac{g(x) \cdot y_2}{\omega} dx = - \int \frac{\frac{e^x}{x^2+1} \cdot x e^x}{e^{2x}} dx$$

$$v_1 = - \frac{1}{2} \int \frac{2x}{x^2+1} dx = \underbrace{- \frac{1}{2} \ln(x^2+1)}_{v_1}$$

$$v_2 = \int \frac{g(x) y_1}{\omega} dx = \int \frac{\frac{e^x}{x^2+1} \cdot e^x}{e^{2x}} dx$$

$$= \int \frac{e^{2x}}{x^2+1} dx = \underbrace{\tan^{-1} x}_{v_2}$$

$$(4) y_p = e^x \cdot \frac{1}{2} \ln(x^2+1) + x e^x + \tan^{-1} x$$

$$(5) g.s = c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln(x^2+1) + x e^x + \tan^{-1} x$$

Ex 8 Find the general solution for DE

$$y'' + 4y = \tan(2x)$$

Variation of parameter

① homog: $y'' + 4y = 0$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_1 = \cos 2x$$

$$y_2 = \sin 2x$$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

② $w[y_1, y_2] =$

$$\begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix}$$

$$= 2 \cos^2 2x + 2 \sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2 \times 1 = 2$$

③ $g(x) = \frac{\tan(2x)}{1} = \tan(2x)$

$$v_1 = - \int \frac{g(x) y_2}{w} dx = - \int \frac{\tan 2x \cdot \sin 2x}{2} dx =$$

$$= - \frac{1}{2} \left[\ln |\sec 2x + \tan 2x| - \frac{\sin 2x}{2} \right]$$

$$v_2 = \int \frac{g(x) y_1}{w} dx = \int \frac{\tan 2x \cos 2x}{2} dx = \frac{1}{4} \cos 2x$$

④ $y_p = y_1 v_1 + y_2 v_2 = \cos(2x) \left[-\frac{1}{4} \ln |\sec 2x + \tan 2x| + \frac{1}{4} \sin 2x \right]$

$$= -\frac{1}{4} \cos(2x) \ln |\sec 2x + \tan 2x| + \frac{1}{4} \sin 2x \cos 2x$$

Let $L[y](x) = y'' + p(x)y' + q(x)y$ and $\{x^{-6}, x^{-1}\}$ be a fundamental solution set to the DE $L[y](x) = 0$.

If $y_p = x^{-6}v_1 + x^{-1}v_2$ is a particular sol for DE $L[y](x) = \frac{e^x}{x^2}$ Find v_2

s.t. $y_1 = x^{-6}$ $y_2 = x^{-1}$

$\otimes W[y_1, y_2] = \begin{vmatrix} x^{-6} & x^{-1} \\ -6x^{-7} & -x^{-2} \end{vmatrix} = -x^{-8} + 6x^{-8} = 5x^{-8}$

$\otimes q(x) = \frac{e^x}{x^2}$

$v_2 = \int \frac{q(x) \cdot y_1}{W} dx = \int \frac{\frac{e^x}{x^2} \cdot x^{-6}}{5x^{-8}} dx = \frac{1}{5} e^x$

Theorem 8 If y_1 is a solution for DE $L[y](x) = ay'' + by' + cy = p_1(x)$ and y_2 is a solution for DE $L[y](x) = ay'' + by' + cy = p_2(x)$

then $y = C_1 y_1 + C_2 y_2$ is a solution for DE

$L[y](x) = ay'' + by' + cy = C_1 p_1(x) + C_2 p_2(x)$

Ex: If $y_1 = \frac{3}{2}x - \frac{9}{4}$ is a sol for DE:

$$y'' + 3y' + 2y = 3x \quad f_1(x)$$

and $y_2 = \frac{e^{3x}}{2}$ is a sol for DE: $y'' + 3y' + 2y = 10e^{3x} \quad f_2(x)$

Find sol for DE: $y'' + 3y' + 2y = -7x + 20e^{3x}$

$f(x)$

$$f(x) = C_1 f_1(x) + C_2 f_2(x)$$

$$C_1 = -3$$

$$C_2 = 2$$

$$y = C_1 y_1 + C_2 y_2 = -3\left(\frac{3}{2}x - \frac{9}{4}\right) + 2\left(\frac{e^{3x}}{2}\right)$$

$$= -\frac{9}{2}x + \frac{27}{4} + e^{3x}$$

Ex 8

If y_1 is sol for DE $y'' + x^2 y' + y = 5x + 3$

y_2 is sol for DE $y'' + x^2 y' + y = x^2 - 2x$

Find the sol for DE $y'' + x^2 y' + y = 2x^2 + 11x + 9$

Ex: Find the general sol for

$$y'' - 2y' - 5 = 0$$

$$y'' - 2y' = 5 \quad \text{non-homog.}$$

We use undetermined coefficient :

$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$$r = 0 \quad r = 2 \quad y_1 = 1 \quad y_2 = e^{2x}$$

$$y_h = C_1 + C_2 e^{2x}$$

$$y(x) = 5 + \frac{5}{2} y_p \quad A \cdot x^k = 1 \quad = Ax$$

$$\textcircled{*} y_p' = A \quad y_p'' = 0 \quad \text{put } y_p, y_p', y_p''$$

$$0 - 2A = 5 \quad \boxed{A = -\frac{5}{2}}$$

$$y_p(x) = -\frac{5}{2}x$$

If $y_1 = -x$ and $y_2 = \frac{x^2}{2} - x$ are solutions for the DE

$$x^3 y''' + f(x) y'' - y = g(x)$$

Find $f(x)$ and $g(x)$

⊗ $y_1' = -1$

$y_1'' = 0$

$y_1''' = 0$

put in DE $\rightarrow x^3 y''' + f(x) y'' - y = g(x)$

$$x = g(x)$$

⊗ $y_2 = \frac{x^2}{2} - x$ $y_2' = x$ $y_2'' = 1$ $y_2''' = 0$

put in DE $\rightarrow x^3 y_2''' + f(x) y_2'' - y_2 = g(x)$

$$f(x) = \left(\frac{x^2}{2} - x \right) = x$$

$$f(x) = x + \frac{x^2}{2} - x = \frac{x^2}{2}$$

Use the undetermined coefficients to find a form of particular solution for $y^{(3)} + 6y'' + 9y' + 4y = e^{-x} + 3^x$

$$r^3 + 6r^2 + 9r + 4 = 0$$

r^3	r^2	r	const
1	6	9	4
	-1	-5	-4
		4	0

$$r = -1$$

$$r^2 + 5r + 4 = 0$$

$$(r+4)(r+1) = 0$$

$$r_2 = -4 \quad r_3 = -1$$

$$y_1 = e^{-x}$$

$$y_2 = x e^{-x}$$

$$y_3 = e^{-x} x$$

$$g(x) = e^{-x} + 3^x$$

$$3^x = e^{x \ln 3}$$

$$e^{-x} + e^{(\ln 3)x}$$

$$= e^{x \ln 3}$$

$$= e^{(\ln 3)x}$$

$$y_p = (A e^{-x})^2 x + (B e^{(\ln 3)x})^2 x$$

$$= A x^2 e^{-x} + B e^{(\ln 3)x}$$

$$y_p = x$$

$$y = \left(x - \frac{e}{2} \right) e^{-x}$$

$$\begin{bmatrix} x \\ 2 \end{bmatrix} = x - \frac{e}{2} x + x = (x - \frac{e}{2})$$

For a unit of time, the relationship between the number of people who are infected and the number of people who are not infected is given by the equation $y = x - \frac{e}{2} x + x = (x - \frac{e}{2})$.

Laplace Transforms :

Def: Let $f(t)$ be a function defined $[0, \infty)$ then the Laplace transform for $f(t)$ is given as :

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) \cdot e^{-st} dt = F(s)$$

provided that the integral converges.

Ex: Find $\mathcal{L}\{3\}$

$$f(t) = 3$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\} = \int_0^{\infty} 3 \cdot e^{-st} dt =$$

$$= \lim_{h \rightarrow \infty} \int_0^h 3e^{-st} dt = \lim_{h \rightarrow \infty} \left[\frac{3e^{-st}}{-s} \right]_0^h =$$

$$= \frac{-3}{s} \lim_{h \rightarrow \infty} [e^{-sh} - e^0] = \boxed{\frac{3}{s}}$$

$f(t)$

$\mathcal{L}\{f(t)\} = F(s)$

Remarks	a	$\frac{a}{s}$
	t^n	$\frac{n!}{s^{n+1}}$
	e^{at}	$\frac{1}{s-a}$
	$\sin(at)$	$\frac{a}{s^2+a^2}$
	$\cos(at)$	$\frac{s}{s^2+a^2}$

Remarque

2.4.7

$$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$$

Exo Evaluate:

$$\boxed{1} \mathcal{L}\{5\} = \frac{5}{s}$$

$$\boxed{2} \mathcal{L}\{\cos(3t)\} = \frac{s}{s^2 + 3^2}$$

$$\boxed{3} \mathcal{L}\{2 \sin(5t) - 4e^{2t} + 10t^5 - 2\}$$

$$= 2 \mathcal{L}\{\sin 5t\} - 4 \mathcal{L}\{e^{2t}\} + 10 \mathcal{L}\{t^5\} - \mathcal{L}\{2\}$$

$$= 2 \left[\frac{5}{s^2 + 5^2} - 4 \cdot \frac{1}{s-2} + 10 \cdot \frac{5!}{s^{5+1}} - \frac{2}{s} \right]$$

$$\boxed{4} \mathcal{L}\{\sinh 3t\} = \mathcal{L}\left\{\frac{e^{3t} - e^{-3t}}{2}\right\} = \frac{1}{2} \mathcal{L}\{e^{3t} - e^{-3t}\}$$

$$= \frac{1}{2} \left[\frac{1}{s-3} - \frac{1}{s+3} \right]$$

5) $\mathcal{L}\{\sin(2t + 2\pi)\} = \frac{1}{s^2 + 4} \cdot \frac{1}{2} = \frac{1}{2(s^2 + 4)}$ أي زاوية تظهر أو تخرج لها (2π) تعود إلى نفس المكان

$$= \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2 + 4} \cdot \frac{1}{2} = \frac{1}{s^2 + 4}$$

$$\{(1,2) \cos - (1,0) \sin\} \mathcal{L} = \{(1,2) \cos (1,0) \sin\} \mathcal{L} \quad (5)$$

Remarks ① $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$

$$\{(1,5) \text{ ② } \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\text{③ } \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\text{④ } \sin \alpha \sin \beta = \frac{1}{2} [\sin(\alpha - \beta) - \sin(\alpha + \beta)]$$

$$\text{⑤ } \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\text{⑥ } \cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha \text{ و } \sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha \quad (6)$$

$$\text{⑦ } \sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha) + \frac{1}{2}$$

$$\text{⑧ } \cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha)$$

Ex: Evaluate

$$\text{① } \mathcal{L}\left\{\sin\left(3t - \frac{\pi}{6}\right)\right\} = \mathcal{L}\left\{\sin 3t \times \cos \frac{\pi}{6} - \cos(3t) \sin\left(\frac{\pi}{6}\right)\right\}$$

$$= \frac{\sqrt{3}}{2} \mathcal{L}\{\sin 3t\} - \frac{1}{2} \mathcal{L}\{\cos 3t\}$$

$$\frac{\sqrt{3}}{2} \cdot \frac{3}{s^2 + 9} - \frac{1}{2} \cdot \frac{s}{s^2 + 9}$$

$$\textcircled{2} \mathcal{L}\{\sin^2(3t)\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(6t)\right\}$$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \times \frac{s}{s^2 + 36}$$

$$\textcircled{3} \mathcal{L}\{\cos(2t)\cos(5t)\} = \mathcal{L}\{\cos(2t) \cdot \cos(5t)\}$$

$$= \mathcal{L}\left\{\frac{1}{2} [\cos(3t) + \cos(7t)]\right\}$$

$$= \frac{1}{2} \left[\frac{s}{s^2 + 9} + \frac{s}{s^2 + 49} \right]$$

$$\textcircled{4} \mathcal{L}\{\sin(3t) \cdot \sin(5t)\}$$

$$\textcircled{5} \int_0^{\infty} e^{-st+10} dt = \int_0^{\infty} e^{10t} \cdot e^{-st} dt = \mathcal{L}\{e^{10t}\} = \frac{1}{s-10}$$

$$\textcircled{6} \int_0^{\infty} e^{-2t} \cos(3t) dt = \mathcal{L}\{\cos(3t)\} \Big|_{s=2}$$

$$= \frac{s}{s^2 + 9} \Big|_{s=2} = \frac{2}{13}$$

Remark 8 shifting property

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\} \Big|_{s \rightarrow s-a}$$

Ex: Evaluate

$$\textcircled{1} \mathcal{L}\{e^{2t} \cos(5t)\} = \mathcal{L}\{\cos(5t)\} \Big|_{s \rightarrow s-2}$$

$$= \frac{s}{s^2 + 25} \Big|_{s \rightarrow s-2} = \frac{s-2}{(s-2)^2 + 25}$$

$$\textcircled{2} \mathcal{L}\{t^3 e^{-4t}\} = \mathcal{L}\{t^3\} \Big|_{s \rightarrow s+4} = \frac{3!}{s^4} \Big|_{s \rightarrow s+4} = \frac{6}{(s+4)^4}$$

$$\textcircled{3} \mathcal{L}\left\{\frac{\sin 3t}{e^{2t}}\right\} = \mathcal{L}\{e^{-2t} \sin 3t\}$$

$$= \mathcal{L}\{\sin 3t\} \Big|_{s \rightarrow s+2} = \frac{3}{s^2 + 9} \Big|_{s \rightarrow s+2}$$

$$= \frac{3}{(s+2)^2 + 9}$$

Remark: $\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n} [\mathcal{L}\{f(t)\}]$

① $\mathcal{L}\{t^2 \cos(2t)\} = (-1)^2 \frac{d^2}{ds^2} [\mathcal{L}\{\cos(2t)\}]$

$$= \frac{d^2}{ds^2} \left[\frac{s}{s^2 + 4} \right] = \frac{d}{ds} \left[\frac{1 + (s^2 + 4) - s + 2s}{(s^2 + 4)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{4 - s^2}{(s^2 + 4)^2} \right] = \frac{-2s \cdot (s^2 + 4)^2 - (4 - s^2) \cdot 2(s^2 + 4) \cdot 2s}{(s^2 + 4)^4}$$

$$= \frac{2s^2 - 24s}{(s^2 + 4)^3}$$

② $\mathcal{L}\{t^{20} e^{3t}\} = \mathcal{L}\{t^{20}\} \Big|_{s \rightarrow s-3} = \frac{20!}{s^{21}} \Big|_{s \rightarrow s-3} = \frac{20!}{(s-3)^{21}}$

③ $\mathcal{L}\{t^4 e^t \cos(3t)\} = \mathcal{L}\{t^4 \cos(3t)\} \Big|_{s \rightarrow s-4}$

$$= \left[(-1)^4 \frac{d^4}{ds^4} \mathcal{L}\{\cos(3t)\} \right] \Big|_{s \rightarrow s-4}$$

$$= \left[-\frac{d}{ds} \left[\frac{s}{s^2 + 9} \right] \right] \Big|_{s \rightarrow s-4}$$

$$= - \frac{s^2 + 9 - s \cdot 2s}{(s^2 + 9)^2} \Big|_{s \rightarrow s-4}$$

$$= \frac{s^2 - 9}{(s^2 + 9)^2} \Big|_{s \rightarrow s-4}$$

$$= \frac{(s-4)^2 - 9}{((s-4)^2 + 9)^2}$$

Invers Laplace Trans Form⁸

Def: If $\mathcal{L}\{f(t)\} = F(s)$, then the invers Laplace is given as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$F(s) = \mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{a}{s}$	a
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2 + a^2}$	$\frac{\sin(at)}{a}$
$\frac{s}{s^2 + a^2}$	$\cos(at)$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$

Remark:

$$\textcircled{1} \mathcal{L}^{-1}\{F(s-a)\} = e^{at} \mathcal{L}^{-1}\{F(s)\}$$

$$\textcircled{2} \mathcal{L}^{-1}\{t^n F(s)\} = \frac{(-1)^n}{n!} \mathcal{L}^{-1}\{F^{(n)}(s)\}$$

$$\textcircled{3} \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{5}{s} \right\} = 5$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+9} \right\} = \frac{\sin(3t)}{3}$$

$$\textcircled{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+16} \right\} = \cos(4t)$$

$$\textcircled{4} \mathcal{L}^{-1} \left\{ \frac{3}{s^9} - \frac{2s}{s^2+4} + \frac{2}{s} + \frac{4}{s-6} \right\}$$

$$= 3 \mathcal{L}^{-1} \left\{ \frac{1}{s^9} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + 4 \mathcal{L}^{-1} \left\{ \frac{1}{s-6} \right\}$$

$$= \frac{3 \times t^8}{8!} - 2 \times \cos 2t + 2 \times 1 + 4 e^{6t}$$

$$\textcircled{5} \mathcal{L}^{-1} \left\{ (s+7)^{-5} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s+7)^5} \right\} = e^{-7t} \times \mathcal{L}^{-1} \left\{ \frac{1}{s^5} \right\}$$

$$= e^{-7t} \times \frac{t^4}{4!}$$

$$\textcircled{6} \mathcal{L}^{-1} \left\{ \frac{7}{(s-3)^2+25} \right\} = 7e^{3t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+25} \right\}$$

$$= 7e^{3t} \frac{\sin 5t}{5}$$

$$\textcircled{7} \mathcal{L}^{-1} \left\{ \frac{s-2}{(s-2)^2+16} \right\} = e^{2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+16} \right\} = e^{2t} \cos(4t)$$

$$\textcircled{8} \mathcal{L}^{-1} \left\{ \frac{s}{(s+3)^2 + 25} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+3-3}{(s+3)^2 + 25} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s+3}{(s+3)^2 + 25} \right\} - \mathcal{L}^{-1} \left\{ \frac{3}{(s+3)^2 + 25} \right\} = e^{-3t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 25} \right\} - 3e^{-3t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 25} \right\}$$

$$= e^{-3t} \cos(5t) - 3e^{-3t} \frac{\sin(5t)}{5}$$

إذا العبارة الترتيبية تحلل : كسور جزئية

$$\textcircled{9} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 - 4s + 5} \right\}$$

إذا العبارة الترتيبية لا تحلل : اكمل مربع

$$= \mathcal{L}^{-1} \left\{ \frac{2}{(s-5)(s+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{\frac{1}{3}}{s-5} - \frac{-\frac{1}{3}}{s+1} \right\}$$

$$= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-5} \right\} - \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

$$= \frac{1}{3} e^{5t} - \frac{1}{3} e^{-t}$$

$$\frac{A}{s-5} + \frac{B}{s+1} = \frac{2}{(s-5)(s+1)}$$

$$A(s+1) + B(s-5) = 2$$

$$A = \frac{1}{3} \quad B = -\frac{1}{3}$$

$$\textcircled{10} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 4s + 3} \right\} =$$

$$\mathcal{L}^{-1} \left\{ \frac{-3}{2} \left[\frac{1}{s+3} - \frac{1}{s+1} \right] \right\}$$

$$= \frac{-3}{2} [e^{-3t} - e^{-t}]$$

$$\frac{3}{s^2 + 4s + 3} = \frac{3}{(s+1)(s+3)}$$

$$\frac{3}{(s+1)(s+3)} = \frac{3}{2} \left[\frac{s+1-s+3}{(s+1)(s+3)} \right]$$

$$= \frac{3}{2} \left[\frac{s+1}{(s+1)(s+3)} - \frac{s+3}{(s+1)(s+3)} \right]$$

$$(11) \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4s + 13} \right\}$$

$$2 \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2 + 9} \right\}$$

$$= 2e^{-2t} \cdot \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 9} \right\}$$

$$= 2e^{-2t} \cdot \frac{\sin 3t}{3}$$

$$s^2 + 4s + 13$$

$$4 = (s+2)^2 - 2$$

$$s^2 + 4s + 4 - 4 + 13$$

$$= (s+2)^2 + 9$$

$$(12) \mathcal{L}^{-1} \left\{ \frac{2}{(s^2+9)(s^2+4)} \right\} = 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+9)(s^2+4)} \right\}$$

$$= \frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{(s^2+9) - (s^2+4)}{(s^2+9)(s^2+4)} \right\} = \frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{s^2+9}{(s^2+9)(s^2+4)} - \frac{s^2+4}{(s^2+9)(s^2+4)} \right\}$$

$$= \frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} - \frac{1}{s^2+9} \right\} = \frac{2}{5} \left[\frac{\sin 2t}{2} - \frac{\sin 3t}{3} \right]$$

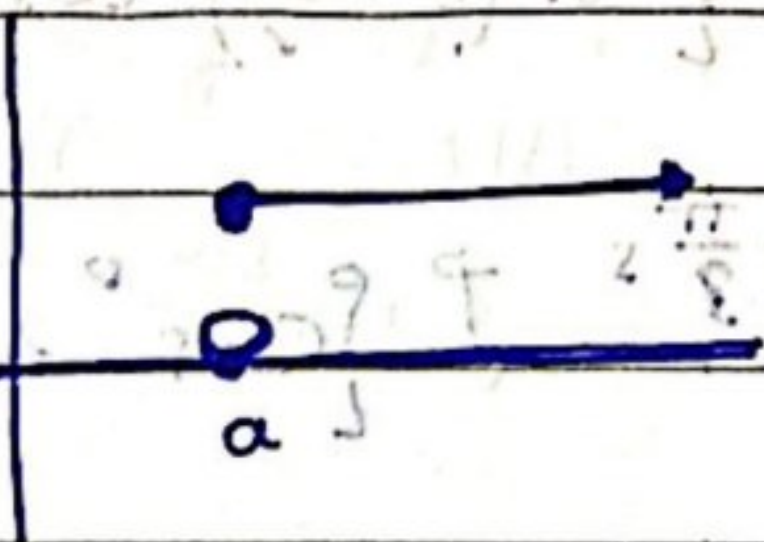
$$(13) \mathcal{L}^{-1} \left\{ \ln \frac{(s+4)}{(s+5)} \right\} = \mathcal{L}^{-1} \left\{ \ln(s+4) - \ln(s+5) \right\}$$

$$= \frac{1}{t} \mathcal{L}^{-1} \left\{ \left[\ln(s+4) - \ln(s+5) \right] \right\} = \frac{1}{t} \mathcal{L}^{-1} \left\{ \frac{1}{s+4} - \frac{1}{s+5} \right\}$$

$$= \frac{1}{t} [e^{-4t} - e^{-5t}]$$

Unit-Step Function :

$$u(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$



Remark :

$$\textcircled{1} \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\textcircled{2} \mathcal{L}\{u(t-a)f(t)\} = e^{-as} \mathcal{L}\{f(t+a)\}$$

$$\textcircled{3} \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a) \cdot \mathcal{L}^{-1}\{F(s)\} \quad t \rightarrow t-a$$

Ex:

$$\textcircled{1} \mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$$

$$\textcircled{2} \mathcal{L}\{u(t-3)\} = \frac{e^{-3s}}{s}$$

$$\textcircled{3} \mathcal{L}\{u(t-5)(t^2+t+2)\} = e^{-5s} \mathcal{L}\{(t+5)^2 + (t+5) + 2\}$$

$$e^{-5s} \mathcal{L}\{t^2 + 10t + 25 + t + 5 + 2\} = e^{-5s} \mathcal{L}\{t^2 + 11t + 32\}$$

$$= e^{-5s} \left(\frac{2!}{s^3} + \frac{11}{s^2} + \frac{32}{s} \right)$$

$$(4) \mathcal{L}\{u(t - \frac{\pi}{2}) \cos(2t)\} = e^{-\frac{\pi}{2}s} \mathcal{L}\{\cos(t + \frac{\pi}{2})\}$$

$$= e^{-\frac{\pi}{2}s} \mathcal{L}\{\cos 2t + \frac{2\pi}{2}\}$$

$$= e^{-\frac{\pi}{2}s} \mathcal{L}\{\cos 2t \cdot \cos \frac{2\pi}{2} - \sin 2t \cdot \sin \frac{2\pi}{2}\}$$

$$= e^{-\frac{\pi}{2}s} \left(\frac{1}{2} \mathcal{L}\{\cos 2t\} - \frac{\sqrt{3}}{2} \mathcal{L}\{\sin 2t\} \right)$$

$$= e^{-\frac{\pi}{2}s} \left(\frac{1}{2} \frac{s}{s^2 + 4} - \frac{\sqrt{3}}{2} \frac{2}{s^2 + 4} \right)$$

$$(5) \mathcal{L}\{u(t-2)e^t\} = \mathcal{L}\{u(t-2)\} \Big|_{s \rightarrow s-3}$$

$$= \frac{e^{-2s}}{s} \Big|_{s \rightarrow s-3} = \frac{e^{-2(s-3)}}{s-3}$$

$$(6) \text{ If } g(t) = \begin{cases} t & t < 2 \\ e^{-3t} & 2 < t < 4 \\ 0 & 4 < t \end{cases} \quad \text{Find } \mathcal{L}\{g(t)\}$$

بجواب تكونه اول فترة t اقل من 2

$$g(t) = t + u(t-2) \cdot (3-t) + u(t-4) \cdot (e^{-3t} - 3)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{u(t-2)(3-t)\} + \mathcal{L}\{u(t-4)(e^{-3t} - 3)\}$$

$$\frac{1}{s^2} + e^{-2s} \mathcal{L}\{3 - (t+2)\} + e^{-4t} \mathcal{L}\{e^{-3(t-4)} - 3\}$$

$$= \frac{1}{s^2} + e^{-2s} \mathcal{L}\{1-t\} + e^{-4t} \mathcal{L}\{e^{3t} \cdot e^{-12} - 3\}$$

$$\frac{1}{s^2} + e^{-2s} \left(\frac{1}{s} - \frac{1}{s^2} \right) + e^{-4t} \left(e^{-12} \cdot \frac{1}{s+3} - \frac{3}{s} \right)$$

Ex-10) If $g(t) = \begin{cases} t & 3 < t < 5 \\ 2 & 5 < t \end{cases}$ Find $\mathcal{L}\{g(t)\}$

$$g(t) = \begin{cases} t & 3 < t < 5 \\ 2 & 5 < t \end{cases}$$

$$g(t) = 0 + u(t-3) \cdot (t-3) + u(t-5) \cdot (2-t)$$

$$= u(t-3)t + u(t-5)(2-t)$$

$$\mathcal{L} u(t-3)t + \mathcal{L} u(t-5)(2-t) = e^{-3s} \mathcal{L}\{t+3\} + e^{-5s} \mathcal{L}\{2-t+5\}$$

$$= e^{-3s} \mathcal{L}\{t+3\} + e^{-5s} \mathcal{L}\{5-3-t\}$$

$$e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right) + e^{-5s} \left(\frac{-3}{s} - \frac{1}{s^2} \right)$$

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s} \right\} = u(t-3) \left\{ 1 + 0 - 0 \right\} = 1$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^4} - \frac{e^{-4s}}{s^2+9} + \frac{3e^{-3s}}{5e^{3s}} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^4} \right\} - \mathcal{L}^{-1} \left\{ \frac{e^{-4s}}{s^2+9} \right\} + 3 \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s} \right\}$$

$$= u(t-2) \mathcal{L}^{-1} \left\{ \frac{1}{s^4} \right\} - u(t-4) \mathcal{L}^{-1} \left\{ \frac{1}{s^2+9} \right\} + 3 u(t-3)$$

$$= u(t-2) \cdot \frac{t^3}{3} - u(t-4) \cdot \frac{\sin 3t}{3} + 3 u(t-3)$$

$$u(t-2) \cdot \frac{(t-2)^3}{3} - u(t-4) \cdot \frac{\sin 3(t-4)}{3} + 3 u(t-3)$$

$$\textcircled{3} \mathcal{L}^{-1} \left\{ \frac{(s-1)e^{-2s}}{(s-1)^2+16} \right\} = u(t-2) \left[\mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+16} \right\} \right]_{t \rightarrow t-2}$$

$$= u(t-2) \left[e^t \mathcal{L}^{-1} \left\{ \frac{s}{s^2+16} \right\} \right]_{t \rightarrow t-2}$$

$$= u(t-2) [e^t \cos 4t]_{t \rightarrow t-2}$$

$$= u(t-2) [e^{t-2} \cdot \cos 4(t-2)]$$

$$\left(\frac{1}{2} - \frac{1}{2} \right) + \left(\frac{1}{2} + \frac{1}{2} \right)$$

Solving the IVP by Laplace transforms

Remark 8

$$E = \frac{1}{2} \text{ and } \left[\left\{ \frac{1}{(1-x)^2} \right\}^{-1} \right] (1-x) \text{ and } \dots$$

$$\textcircled{1} \mathcal{L} \{ y^{(n)}(t) \} = s^n \mathcal{L} \{ y \} - [s y(0) - s y'(0) + \dots + y^{(n-1)}(0)]$$

If $n=1$

$$\frac{[P(9-4) \cdot 6 \cdot 8 \cdot 7] (40-4)}{L} =$$

$$\mathcal{L}\{y'(t)\} = s \mathcal{L}\{y\} - y(0)$$

To solve an IVP by Laplace:

$$\left\{ \begin{array}{l} 18 + 2(2-2) \end{array} \right\} \cdot 1 \text{ (1)}$$

① Take L for both sides of DE

$$\frac{1}{(15x^2)(x+2)} = \frac{A}{15x^2} + \frac{B}{x+2}$$

② Find $\mathcal{L}\{y(t)\} = Y(s)$

③ Take I^{-1} to both sides in part ② = (1P. 1T) ③

Solve the IVP by LT:

$$y'' - 6y' + 13y = 0 \quad y(0) = 0 \quad y'(0) = -3$$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = 0$$

$$s^2 \mathcal{L}\{y\} - s y(0) - y'(0) - 6(s \mathcal{L}\{y\} - y(0)) + 13\mathcal{L}\{y\} = 0$$

$$= s^2 \mathcal{L}\{y\} + 3 - 6s \mathcal{L}\{y\} + 13\mathcal{L}\{y\} = 0$$

$$(s^2 - 6s + 13) \mathcal{L}\{y\} = -3 \Rightarrow \mathcal{L}\{y\} = \frac{-3}{s^2 - 6s + 13} = Y(s)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-3}{s^2 - 6s + 13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-3}{(s-3)^2 + 4}\right\}$$

$$= -3 e^{3t} \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\}$$

$$= -3 e^{3t} \frac{\sin 2t}{2} = \frac{-3}{2} e^{3t} \sin(2t)$$

Use Laplace transform to solve the IVP. $y'' - 4y' + 4y = t^3 e^{2t}$, $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$s^2 \mathcal{L}\{y\} - sy(0) - y'(0) - 4s \mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3\}$$

$$s^2 \mathcal{L}\{y\} - 4s \mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{3!}{s^4} \quad | \quad s \rightarrow s-2$$

$$(s^2 - 4s + 4) \mathcal{L}\{y\} = \frac{6}{(s-2)^4}$$

$$\mathcal{L}\{y\} = \frac{6}{(s-2)^4 (s^2 - 4s + 4)} = Y(s)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^4 (s^2 - 4s + 4)}\right\}$$

$$= 6 \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^6}\right\} = 6e^{2t} \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = 6e^{2t} \cdot \frac{t^5}{5!} = \frac{6e^{2t} t^5}{120}$$

Solve the IVP by Laplace transform

$$y' - 2y = g(t), \quad y(0) = 0 \quad g(t) = \begin{cases} 0 & t < 1 \\ 2 & t > 1 \end{cases}$$

$$g(t) = 0 + u(t-1) \cdot (2-0) = 2u(t-1)$$

$$y' - 2y = 2u(t-1)$$

$$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 2\mathcal{L}\{u(t-1)\}$$

$$s\mathcal{L}\{y\} - y(0) - 2\mathcal{L}\{y\} = 2\frac{e^{-s}}{s}$$

$$s\mathcal{L}\{y\} - 2\mathcal{L}\{y\} = 2\frac{e^{-s}}{s}$$

$$(s-2)\mathcal{L}\{y\} = 2\frac{e^{-s}}{s} \Rightarrow \mathcal{L}\{y\} = \frac{2e^{-s}}{(s-2)s} = Y(s)$$

$$\mathcal{L}^{-1}\left\{\frac{2e^{-s}}{(s-2)s}\right\} = 2u(t-1) \left[\mathcal{L}^{-1}\left\{\frac{1}{(s-2)s}\right\} \right]_{t \rightarrow t-1}$$

كسور جزئية

$$2u(t-1) \left[\mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{-1/2}{s}\right\} \right]_{t \rightarrow t-1}$$

$$2u(t-1) \left[\frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \right]_{t \rightarrow t-1}$$

$$= 2u(t-1) \left[\frac{1}{2} e^{2t} - \frac{1}{2} \cdot 1 \right]_{t \rightarrow t-1}$$

$$2u(t-1) \left[\frac{1}{2} e^{2(t-1)} - \frac{1}{2} \right] = u(t-1)(e^{2t-2} - 1)$$

$$y'' + p_1 y' - 2y = -2 \text{ and } y(0) = 0 \text{ and } y'(0) = 0$$

Series Solution of D.E.s

Def ① A Taylor series of a function $f(x)$ about $x = x_0$

$$\text{is } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

② A Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

If $p_1(x), \dots, p_n(x), g(x)$ are continuous at x_0 and $y(x)$ is a function that has derivatives of all orders at x_0 any $y(x)$ is solution for the IVP

$$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = g(x) \text{ where}$$

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

Then y has a power series expansion in the form

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!} (x - x_0)^2 + \dots$$

Find at least the 5th non zero terms in a power series expansion of the solution of TVP

$$y'' + y' - xy = 3 \quad y(0) = 1 \quad y'(0) = -2$$

$$x_0 = 0 \quad \boxed{y(0) = 1} \quad \boxed{y'(0) = -2}$$

$$y'' + y' - xy = 3 \quad \text{put } x=0 \quad \Rightarrow \quad y''(0) + y'(0) = 3$$

$$y''(0) + y'(0) = 3 \Rightarrow y''(0) - 2 = 3 \Rightarrow y''(0) = 5$$

$y''' + y'' - y - xy' = 0$ $y'''(0) + 5 - 1 - 0 = 0$ $\boxed{y'''(0) = -4}$	$y^{(4)} + y''' - y' - y - xy'' = 0$ $y^{(4)}(0) + -4 + 2 + 2 - 0 = 0$ $\boxed{y^{(4)}(0) = 0}$
---	---

$$y^{(5)} + y^{(4)} - y'' - y''' - xy'' = 0$$

$$y^{(5)}(0) + 0 - 5 - 5 - 0 - 5 = 0 \Rightarrow \boxed{y^{(5)}(0) = 15}$$

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5$$

$$1 - 2x + \frac{5}{2}x^2 - \frac{4}{6}x^3 + 0 + \frac{15}{120}x^5$$

~~Find~~ Find the 1st 4 - non zero terms in a power series expansion of the solution to IVP

$$y'' - xy = 5 + 6x + 3x^3 \quad y(-1) = 1 \quad y'(-1) = 0$$

$$x_0 = -1 \quad y(-1) = 1$$

$$x = -1 \text{ in DE}$$

$$y''(-1) - (-1)y(-1) = 5 + 6(-1) + 3 \cdot 0 = 0$$

$$y''(-1) + 1 = 5 - 6 - 3 \rightarrow y''(-1) = -5$$

$$y''' - y - xy' = 6 + 9x^2$$

$$y''' - 1 + 0 = 6 + 9 \rightarrow y''' = 16$$

$$y^{(4)} - y' - y' - xy'' = 18x$$

$$y^{(4)} = -13$$

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$= y(-1) + \frac{y'(-1)(x+1)}{1!} + \frac{y''(-1)(x+1)^2}{2!} + \frac{y'''(-1)(x+1)^3}{3!} + \frac{y^{(4)}(-1)(x+1)^4}{4!} + \dots$$

$$1 + 0 + \frac{-5}{2!}(x+1)^2 + \frac{16}{6}(x+1)^3 - \frac{13}{24}(x+1)^4 + \dots$$

Solve the DE (using series)

$$y'' - y = 0 \quad \text{about } x=1$$

$$x_0 = 1$$

$$y(1) = a$$

$$y'(1) = b$$

$$y'' - y = 0 \quad \text{put } x=1$$

$$y''(1) - y(1) = 0 \rightarrow y''(1) = a$$

$$y'''(1) - y'(1) = 0 \rightarrow y'''(1) = b$$

$$y^{(4)}(1) - y''(1) = 0 \rightarrow y^{(4)}(1) = a$$

$$y^{(5)}(1) - y'''(1) = 0 \rightarrow y^{(5)}(1) = b$$

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x-x_0)^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!} (x-1)^n$$

$$y(1) + \frac{y'(1)}{1!} (x-1)^1 + \frac{y''(1)}{2!} (x-1)^2 + \frac{y'''(1)}{3!} (x-1)^3 + \frac{y^{(4)}(1)}{4!} (x-1)^4 + \frac{y^{(5)}(1)}{5!} (x-1)^5 + \dots$$

$$a + b(x-1) + \frac{a}{2!} (x-1)^2 + \frac{b}{3!} (x-1)^3 + \frac{a}{4!} (x-1)^4 + \frac{b}{5!} (x-1)^5 + \dots$$

$$a \left(1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} + \dots \right) + b \left((x-1) + \frac{(x-1)^3}{3!} + \frac{(x-1)^5}{5!} + \dots \right)$$

$$a \sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{2^n n!} + b \sum_{n=0}^{\infty} \frac{(x-1)^{2n+1}}{(2n+1)!}$$

H.W.8 Find the First five non-zero terms in a power series expansion at the solution to DE :

$$\textcircled{1} y'' - xy = 5 + 6x - 2x^3 \quad y(0) = 1, y'(0) = 0$$

$$\textcircled{2} y'' - xy = 5 + 6x - 2x^3 \quad \text{about } x = -1$$

Ex.8 use Laplace transform to solve the IVP :

$$y'' + t y' - 2y = -2 \quad y(0) = 0, y'(0) = 0$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{t y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{-2\} \quad \left| \begin{array}{l} \mathcal{L}\{t y'\} \\ = (-1)' \frac{d}{ds} [\mathcal{L}\{y'\}] \end{array} \right.$$

$$s^2 \mathcal{L}\{y\} - s y(0) - y'(0) - [s \mathcal{L}\{y'\}]' - 2\mathcal{L}\{y\} = \frac{-2}{s}$$

$$s^2 \mathcal{L}\{y\} - [s \mathcal{L}\{y\} - y(0)]' - 2\mathcal{L}\{y\} = \frac{-2}{s} \quad \mathcal{L}\{y\} = Y(s)$$

$$s^2 Y(s) - [s \cdot Y(s)]' - 2 Y(s) = \frac{-2}{s}$$

$$s^2 Y(s) - Y - s Y' - 2 Y = \frac{-2}{s}$$

$$(s^2 - 3)y - sy' = \frac{-2}{s}$$

$$sy' - (s^2 - 3)y = \frac{2}{s} \rightarrow \text{Linear in } y$$

$$y' - \frac{(s^2 - 3)}{s}y = \frac{2}{s^2}$$

$p(s) = -\frac{(s^2 - 3)}{s} = -s + \frac{3}{s}$	$q(s) = \frac{2}{s^2}$
--	------------------------

$$\mu(s) = e^{\int -s + \frac{3}{s} ds} = e^{-\frac{s^2}{2} + 3 \ln(s)} = e^{-\frac{s^2}{2}} s^3$$

$$y(s) = \frac{1}{s^3} \int \frac{s^3}{e^{-\frac{s^2}{2}}} \cdot \frac{2}{s^2} ds + \frac{C e^{-\frac{s^2}{2}}}{s^3}$$

$$= \frac{e^{-\frac{s^2}{2}}}{s^3} \int 2s e^{\frac{s^2}{2}} ds + \frac{C e^{-\frac{s^2}{2}}}{s^3}$$

$$= \frac{e^{-\frac{s^2}{2}}}{s^3} \left[-2e^{\frac{s^2}{2}} s \right] + \frac{C e^{-\frac{s^2}{2}}}{s^3}$$

(2) $y(s) = \frac{-2}{s^3} + C \frac{e^{-\frac{s^2}{2}}}{s^3}$	$\lim_{s \rightarrow \infty} y(s) = 0$ منه يكون $C=0$ بجانب تكون $C=0$
--	--

$$y(t) = \mathcal{L}^{-1}(s) = \mathcal{L}^{-1}\left\{\frac{-2}{s^3}\right\}$$

$$= -2 \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = -2 * \frac{t^2}{2!} = \boxed{-t^2}$$