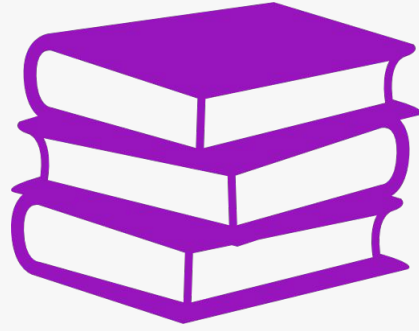




TurbolEG.com



دفتر:

تفاضل و تكامل 1 Calculus 1

إعداد : بسنت سامح للدكتورة : آلاء القدومي

اللجنة الأكاديمية لقسم الهندسة الصناعية

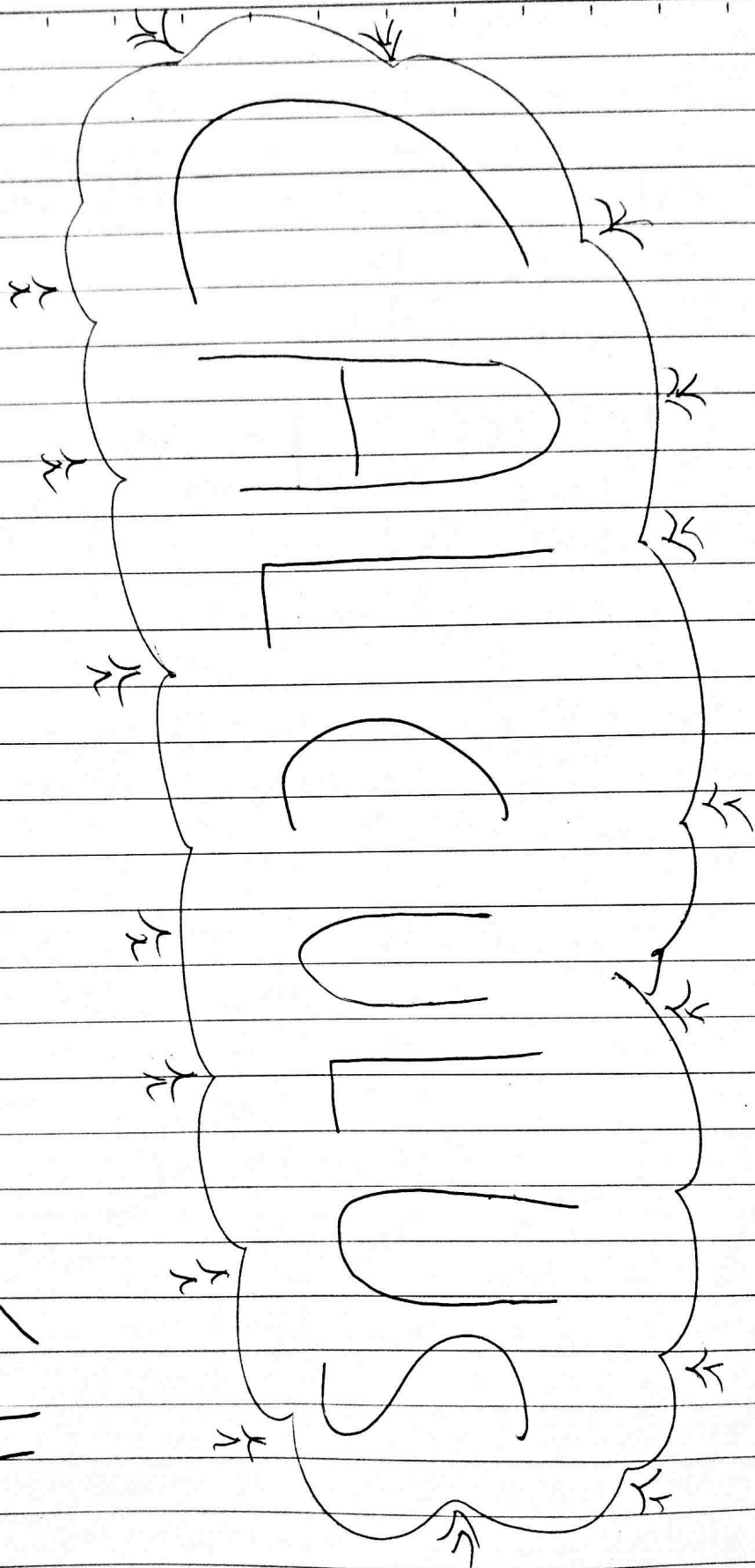
2023



جوجل

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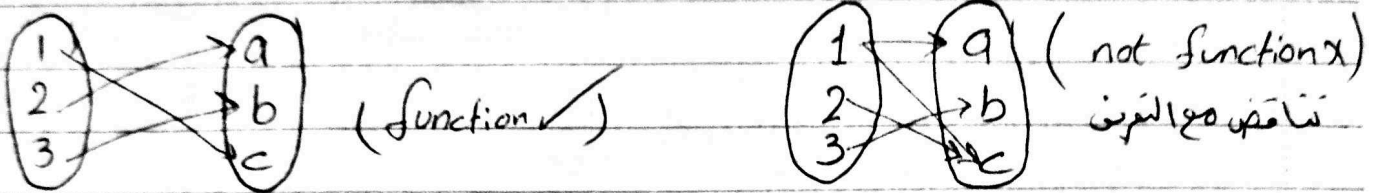
$D_s = IR$ كشاكشور
 $D_s = ID$ $R_s = \frac{1}{C} \int C$ C
 $D_s IR_f = IR$ اقتراح خط
 $D_s = IR$, $R_s = (F(1 - \frac{b}{2a}))$ و ∞ اقتراح الترتيب

$\sqrt{x}$ $\sqrt[3]{x}$ $\sqrt{x}$ $\frac{1}{x}$

Lecture 1

* الأفتان :- Function *

علاقة تربط كل عنصر بالمجال بعنصر واحد فقط بالمجال الآخر :- الأفتان function
 * يوجد لدينا مجموعتين واحدة مجال و أخرى مدى إذا ارتبط بعنصر بالمجال بعنصر واحد فقط
 بالمدى يسمى العلاقة أفتان وإذا ارتبط بأكثر لا يسمى أفتان كل



العنصر (1) ارتبط بـ (a) and (c) (عنصرين) :- حقيقة الشرط تنفي أفتان

وكتابة قاعدة الأفتان على شكل
 Domain = مجال = input (مدخلات) = x
 Range = مدى = output (مخرجات) = y
 up = ق (س) و $y = f(x)$
 ex: $f(x) = \sqrt{x+2}$ و $v = \sqrt{x+2}$ ق (س)

Ex: $f(x) = \sqrt{x+2}$

① $f(2) = \sqrt{2+2} = \sqrt{4} = 2$ نفرض مكان كل x القيمة (2)

② $f(-4) = \sqrt{-4+2} = \sqrt{-2} \Rightarrow \text{undefined}$

undefined (نفا) قيمة غير معروفة لا يجوز وجود الـ تحت الجذر التربيعي يجب ان يكون موجب

③ $f(m+3) = \sqrt{m+3+2} = \sqrt{m+5}$ فرضه m عدد صحيح موجب

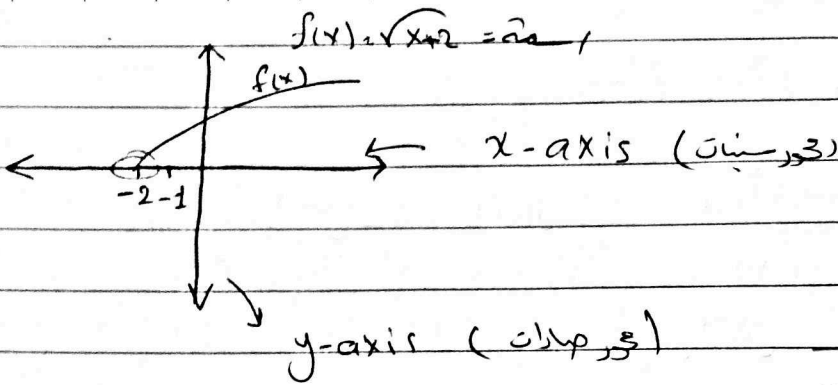
④ Domain f = $[-2, \infty)$ قيم (قيم x المسموح نضعها داخل الأفتان) اختبار

* بما ان الجذر تربيعي يجب ان يكون ما بداخله موجب او صفر لان $0 \leq \sqrt{\quad}$ إذا $x = -2$ $\sqrt{-2+2} = 0$

لو وضعنا $x = -2.1$ x خطا لان سينتج قيمة سالبة. undefined اي $x = -5, x = -2$

* نأخذ قيم المجال التي تجعل للأفتان صحيح Domain:

نقطة صفر $f(-1)=0$ $\sqrt{-2+2}=0$



نقطة صفر:

لا يملك المجال صفر الرسم
تسقط رسمه الاقتران على
محاور السينات

كل نقطة على رسم $f(x)$ $\rightarrow$ نقطة صفر
تحتوي نقطة صفر السينات وهو المجال
(-2, 0)

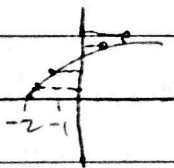
مخرجان نتيجة تعريف المجال

⑤ Range $f = [0, \infty)$ مخرج $\leftarrow \begin{matrix} 0 \\ \sqrt{-2+2} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$
Domain $f = [-2, \infty)$ $\rightarrow \sqrt{-2+2}=0$

نقطة صفر يكون في نقطة تحول تجعل الأعداد تقدر ان يكون واضح ان الأعداد تنزله

ما هو عدد المجال صفر الاقتران بهذا صفرية باقتران $\sqrt{x+2}$

* ملاحظة 1 -



كل نقطة على f تحتوي

نقطة على محور اليعات

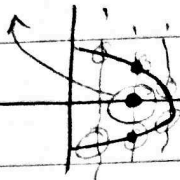
لا يملك الرسم نقطة صفر الاقتران

على محور اليعات

* ملاحظة 2 - المجال والمخرج هو ما يحدد الاقتران على اسقاط الاقتران على محور السينات
او اليعات

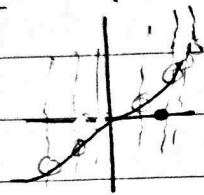
* اختبار الخط العمودي The Vertical line test *

هناك بعض الحالات



* نرسم خطوط عمودية إذا قطع الخط العمودي الاقتران
بأكثر من نقطة $\Rightarrow$ لو كان خط واحد المنحني لا يمثل اقتران
(not function x)

(ج) له صورتين يعني $\Rightarrow$ يرتبط به نقطتين $\Rightarrow$ فإلا تعريف الاقتران الجان يربط بعنصر واحد بالعدد (نظام)
(س) ليست ما ينبغي اقتران ؟



نرسم خطوط عمودية تقطع الاقتران بنقطة واحدة
يمثل اقتران (function ✓)

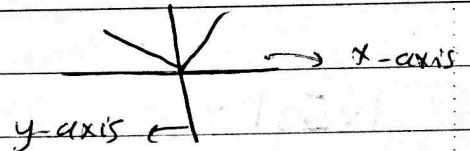
له صورة واحدة كل عدد بالجانب يرتبط بعنصر واحد بالعدد (تحقق) تعريف الاقتران

* ما ينبغي الاشارة اقتران اذا ارتبط عنصره الجان بأكثر من واحد بالعدد (وشرط العكس) فهي

مختلفة

* The absolute value function * اقتران القيمة المطلقة *

$$f(x) = |x| = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$$



$$f(0) = |0| = 0, \quad f(2) = |2| = 2, \quad f(-2) = |-2| = 2$$

مجموع قيم x المسموح بتعريف اعداد حقيقية $\Rightarrow$ Domain = $(-\infty, \infty) = \text{real number} = \mathbb{R}$
داخل هذا الاقتران (هنا كذا اعداد)

* مجال الرسم صنف الرسم على محور السينات (لواقطها لتحجز كل محور السينات)

$$\text{Range} = [0, \infty)$$

* الصور التي تنتج من تعريف قيم الجان

* من الرسم صنف الرسم على محور الصادات لتحجز $[0, \infty)$

① $|ab| = |a||b|$

* خاصية المثلث

② $|a/b| = |a|/|b|$

③ $\sqrt{x^2} = |x|$

④ $(\sqrt{x})^2 = x$

* ⑤ $|x| = a \rightarrow x = -a \text{ or } x = a$

* ⑥ $|x| \leq a \rightarrow -a \leq x \leq a$ إذا في a هو ما في ماواة

⑦ $|x| \geq a \rightarrow x \geq a \text{ or } x \leq -a$

(قيمة)

Example : Find the Value(s) of x

① $|x| = 3 \rightarrow x = 3 \text{ or } x = -3$ هذه هي المعادلة في كلا الاتجاهين

② $|x-2| < 5 \rightarrow -5 < x-2 < 5$ ما في ماواة بالحوال و بنحو ههنا
 $+2 \quad +2 \quad +2$
 $-3 < x < 7$ كل المثلث

أي عدد لو عوضنا به يحقق المثلث ههنا
 $x \in (-3, 7)$ مفتوحة عتات و
 $3 \quad 7$ فتن ماواة

③ $|x-3| \geq 4 \rightarrow x-3 \geq 4 \text{ or } x-3 \leq -4$
 $+3 \quad +3 \quad +3 \quad +3$
 $x \geq 7 \text{ or } x \leq -1$

$x \in (-\infty, -1] \cup [7, \infty)$ الجواب
 $x \geq 7$
 $x \leq -1$

Example : Rewrite

① نأوي ما باخل المثلث بهنفر $\rightarrow$ (إعادة تعريف)

بغير ترتيب

و نغوضه بما باخل المثلث بهنفر

② نأخذ

$f(x) = |x-1|$

$x-1 = 0 \rightarrow x = 1$

③ لو عوضنا ال (1) ههنا ماواة 0

$-(x-1) \quad + \quad (x-1)$

$f(x) = \begin{cases} -(x-1) & x < 1 \\ x-1 & x > 1 \end{cases} \rightarrow \begin{cases} -x+1 & x < 1 \\ x-1 & x > 1 \end{cases}$

0 = 1
 لو عوضنا به ههنا ماواة عتات ماواة
 لو عوضنا به ههنا ماواة عتات ماواة

Homework: $f(x) = |2-x|$
Rewrite

$$2-x=0 \rightarrow \boxed{x=2}$$

$$\begin{array}{c} + \quad - \\ \hline (2-x) \quad 2 - (2-x) \end{array}$$

$$f(x) = \begin{cases} 2-x, & x \leq 2 \\ -2+x, & x \geq 2 \end{cases} \rightarrow \text{or} \text{ } \leftarrow$$

* Polynomials کثیران الحدود * الحاصل عاری یسوی است

س ۸-۶، ۵، ۴، ۳، ۲، ۱، ۰

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0$$

قول کلهم اعداد توان (اعداد حقیقه نسبی) $R = (-\infty, \infty)$

* The number $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are constants called the coefficients of the polynomial. ^{توان} ^{ضرایب} کثیران الحدود

* the degree of the polynomial is (n) ^{درجه} ^{شماره}

* (n) is nonnegative integer ^{صورت} ^{عدد}

(n) is nonnegative integer (غیر منفی) ^{شماره} ^{فوجیه}

* The domain of any polynomial is $R = (-\infty, \infty)$ ^{مجال الاقتران کثیران الحدود (Real num)} ^{الا لوجیه شرط مساوی}

nonnegative ^{غیر منفی} ^{شماره} ^{فوجیه}

integer ^{عدد صحیح}

real number ^{عدد حقیقی}

natural number ^{عدد طبیعی}

Positive ^{موجب}

non Positive ^{غیر موجب}

Example: $f(x) = 3x^5 + 7x - 4 =$

$$3x^5 + 0x^4 + 0x^3 + 0x^2 + 7x - 4$$

کثیر حدود لانه الحاصل کلها اعداد حقیقه وال حشر حشر

$$a_n x^n, a_{n-1} x^{n-1}$$

non Positive ^{غیر موجب} ^{شماره} ^{فوجیه}

$$① f(x) = 3x^5 + 7x - 4$$

Polynomial with degree (5), constant term = -4
 (أكبر درجة) (الـ ثابت)

$$② f(x) = \sqrt{x} + 9x^4 + 1 = x^{1/2} + 9x^4 + 1$$

* البادئ أعداد حاصبة (كأن القوى $(\frac{1}{2})$) غير صحيح

Not polynomial

$$③ f(x) = 9x^{-5} - 4$$

القوى (أعداد غير صحيح)

Not polynomial

$$④ f(x) = 9 = 9x^0$$

قوى غير سالبة $\rightarrow$ موجب
 عدد حقيقي

اقتراح ثابت

Polynomial

$$⑤ f(x) = |x| + 5x + 7 = \begin{cases} -x + 5x + 7, & x \leq 0 \\ x + 5x + 7, & x > 0 \end{cases}$$

$$\therefore f(x) = \begin{cases} 4x + 7, & x \leq 0 \\ 6x + 7, & x > 0 \end{cases}$$

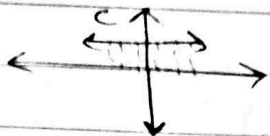
اقتراح متشعب

Not Polynomial

Constant function

دالة ثابتة

$$f(x) = c$$



خط افقي موازي لمحور السينات

العدد حقيقي

$$\text{Domain } f = \mathbb{R} = (-\infty, \infty)$$

$$\text{Range } f = \{c\}$$

العدد حقيقي

Domain

Example 1 $f(x) = 5$, $f(x) = 0$, $f(x) = -1$

لقد درسنا كثير من الدوال في فترات (R) وصورها الصورة الناتجة

والرسم خط موازي لمحور السينات

اقتراحاتي

Linear function

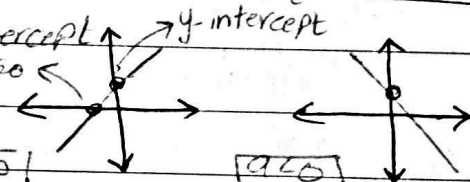
دالة خطية

$(x=0)$

$$y = f(x) = 0$$

X-intercept

Y-intercept



$$f(x) = ax + b, a \neq 0$$

$$\text{Domain } f = \mathbb{R}$$

$$\text{Range } f = \mathbb{R}$$

$$a > 0$$

$$a < 0$$

$$y = f(x) = 0$$

و هو يمثل قطع السيني

$$(x=0)$$

و هو يمثل قطع السيني

(x)

Example :- $f(x) = 2x + 4$

$$0 = y$$

X-intercept

$$y = 0$$

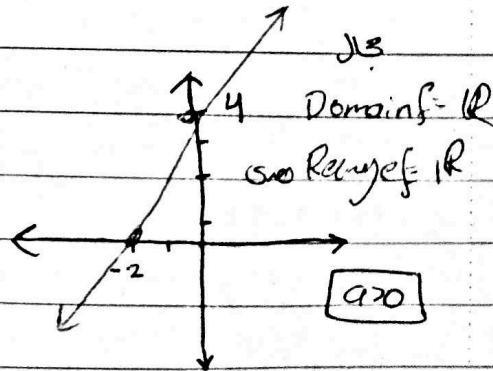
و هو يمثل قطع السيني

$$0 = 2x + 4 \rightarrow 2x = -4 \rightarrow x = -2$$

Y-intercept $(x=0)$

$$f(0) = 2(0) + 4 = 4 \quad a = 2 \quad b = 4$$

عندما $a > 0$ موجبة



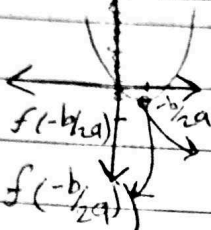
القوس المكافئ Parabola

A Quadratic Function: الاقتران التربيعي *

$f(x) = ax^2 + bx + c, (a \neq 0)$ Domain $f = \mathbb{R}$

لأعلى $a > 0$

If $a > 0$



* رسم الاقتران التربيعي إما للأعلى أو للأسفل

• نجد المعطى أو المعطى نجد الاقتران التربيعي

نقطة قطع المحاور
المكافئ

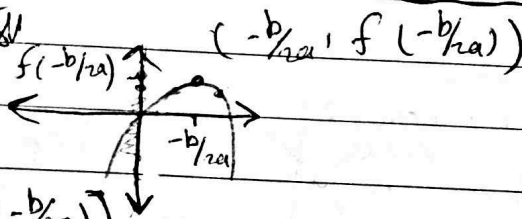
Range $f = [f(-b/2a), \infty)$

• إذا parabola مفتوح للأعلى $a > 0$

نقطة الرأس (y)

$a < 0$

If $a < 0$



• الفترة من الأصغر للأكبر

Range $f = (-\infty, f(-b/2a)]$

Example:-

① $f(x) = x^2 - 6x + 7$

$a=1, b=-6, c=7$

مفتوح للأعلى $a > 0$

Domain $f = \mathbb{R}$

Range $f = [f(-b/2a), \infty) = [f(-(-6)/2(1)), \infty)$

القاعدة 1

$[f(3), \infty) = [3^2 - 6(3) + 7, \infty) = [-2, \infty)$

* $y=0$ $x=0$ $x=7$ $y=0$ $x=0$ $x=7$ $y=0$ $x=0$ $x=7$

$0 = x^2 - 6x + 7$

نقطة تقاطع المحاور

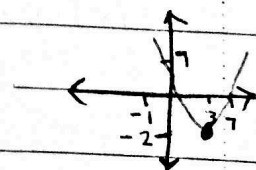
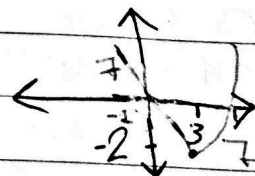
$(x+1)(x-7)$

$x=-1, x=7$

$y=0+0+7$

$y=7$

المقطع الصادي



② $f(x) = 3 + 2x - x^2$ $a = -1 < 0$ $b = 2$ $c = 3$

Domain $f = \mathbb{R}$

Range $f = (-\infty, f(-\frac{b}{2a})] = (-\infty, f(-\frac{2}{2(-1)})] = (-\infty, f(1)]$
 $(-\infty, 3 + 2(1) - 1^2] = (-\infty, 4]$

* Quadratic equation

عبارة تربيعية *

$ax^2 + bx + c = 0$, $a \neq 0$

حل (عبارة عند خطية) $\begin{matrix} > 0 \\ = 0 \\ < 0 \end{matrix}$ $\begin{matrix} \uparrow \\ \uparrow \end{matrix}$ $\begin{matrix} \text{حل واحد} \\ \text{حلين} \end{matrix}$

Discriminant $\Delta = b^2 - 4ac$ $\begin{matrix} \rightarrow \Delta > 0 \\ \rightarrow \Delta < 0 \end{matrix}$ $\begin{matrix} \uparrow \\ \uparrow \end{matrix}$ $\begin{matrix} \text{حل واحد} \\ \text{حلين} \end{matrix}$

التنوين العام

$x = \frac{-b \pm \sqrt{\Delta}}{2a}$

في حال كانت
العبارة التربيعية تملك

Example:- Find the Value(s) of x

① $x^2 + x + 12 = 0$ $a = 1$, $b = 1$, $c = 12$

$\Delta = b^2 - 4ac = 1 - 4(1)(12) = 1 - 48 = -47 < 0$ $\rightarrow$ لا توجد حلول
has no solutions or roots

② $x^2 - 3x - 4 = 0$ $a = 1$, $b = -3$, $c = -4$

$\Delta = (-3)^2 - 4(1)(-4) = 25 > 0$ $\rightarrow$ لها حلين

$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-3) \pm 5}{2} = \frac{3 \pm 5}{2}$

$x = [-1, 4]$

$\frac{3+5}{2} = \frac{8}{2} = 4$
 $\frac{3-5}{2} = \frac{-2}{2} = -1$

$(x+1)(x-4) = 0$ $x^2 - 3x - 4 = 0$ $\rightarrow$ لو طبقنا حل هتبقى $x = -1, 4$

* Power function *

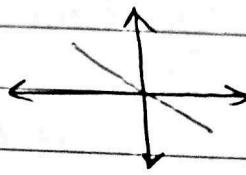
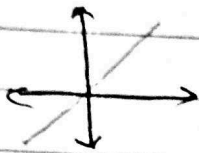
$$f(x) = x^p \quad , \quad p \text{ real number}$$

$f(x) = x^n$, n positive integer

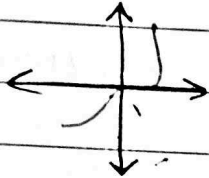
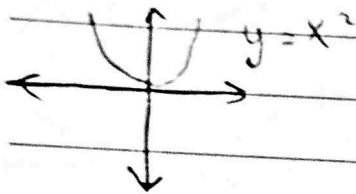
Power functions

طرح موج

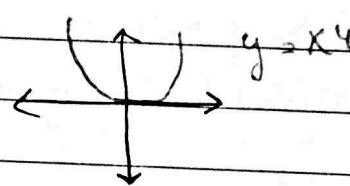
$$f(x) = x^1$$



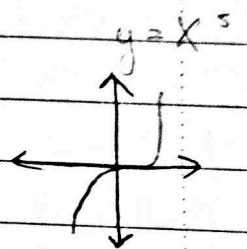
$$f(x) = -x$$



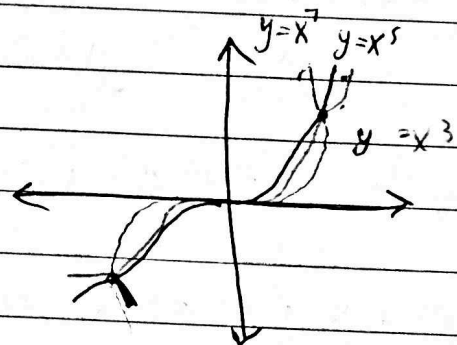
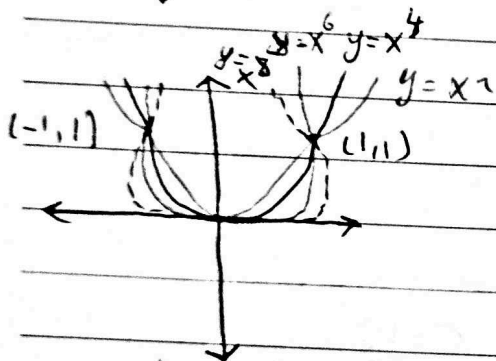
$$y = x^3$$



$$y = x^4$$



$$y = x^5$$



کلیه نمودارهای فوق را در شکل زیر به تفصیل مشاهده کنید

در کتابهای ریاضیات

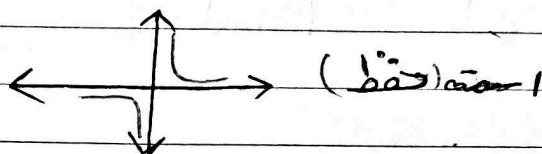
(Rational Function) $\rightarrow$

* Rational Function $\rightarrow$

$$f(x) = \frac{Q(x)}{P(x)} = \frac{\text{Numerator}}{\text{Denominator}}$$

Example :- ① $f(x) = \frac{x^2 + 5x}{x^2 - 6x + 1}$ $\leftarrow$ (Rational Function) $\checkmark$
 $x^2 - 6x + 1 \leftarrow$ (Denominator) $\checkmark$

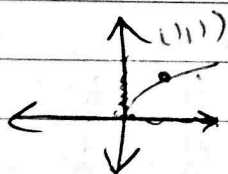
② $f(x) = \frac{1}{x}$



Domain $f = \mathbb{R} \setminus \{0\}$
 Range $f = \mathbb{R} \setminus \{0\}$

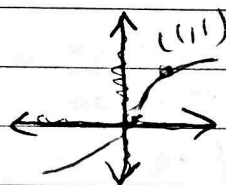
* Root Function *

$$f(x) = x^{1/n} = \sqrt[n]{x}, \quad n \text{ positive integer}$$



$$f(x) = \sqrt{x} = x^{1/2}$$

Domain $f(x) = [0, \infty)$
 Range $f(x) = [0, \infty)$



$$f(x) = \sqrt[3]{x} = x^{1/3}$$

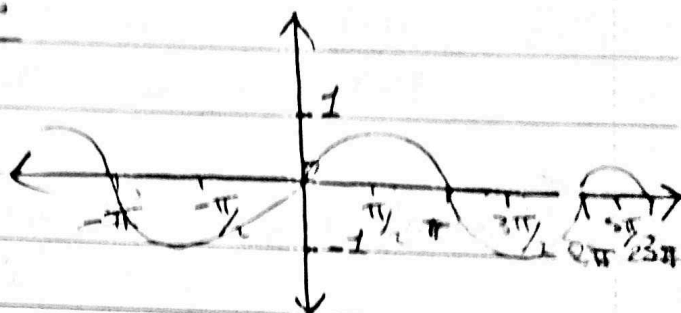
Domain $f(x) = \mathbb{R}$
 Range $f(x) = \mathbb{R}$

* Trigonometric function *

① $f(x) = \sin x$ (جيب) = $\frac{\text{ضلع مقابل}}{\text{فرض}}$

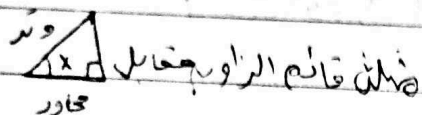
Domain $f = \mathbb{R}$

Range $f = [-1, 1]$



$f(x) = \sin x$

$-1 \leq \sin x \leq 1$, $|\sin x| \leq 1$



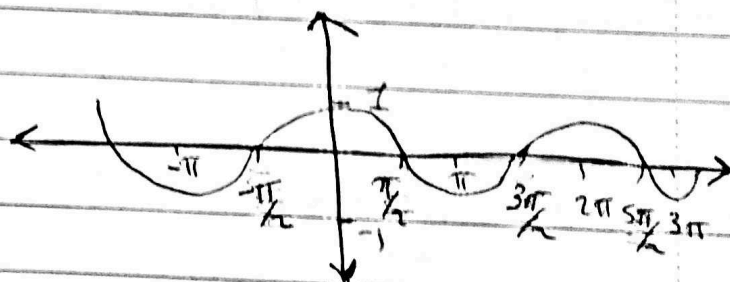
جيب (sin)

② $f(x) = \cos x$ = $\frac{\text{ضلع مجاور}}{\text{فرض}}$

Domain $f = \mathbb{R}$

Range $f = [-1, 1]$

$|\cos x| \leq 1$



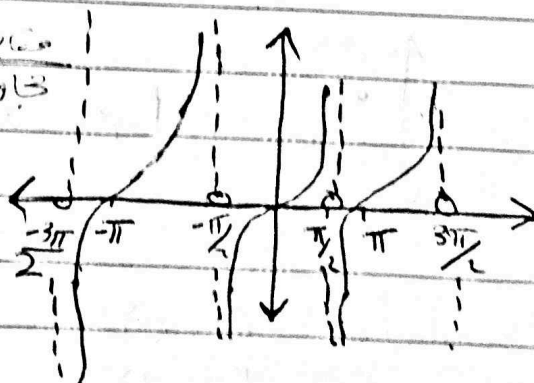
$f(x) = \cos x$

جيب (cos)

③ $f(x) = \tan x = \frac{\sin x}{\cos x} = \frac{\text{ضلع مقابل}}{\text{ضلع مجاور}} = \frac{\text{مقابل}}{\text{جوار}}$

Domain $f = \mathbb{R} \setminus \left\{ \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2} \right\}$

Range $f = \mathbb{R}$



$f(x) = \tan x$

$$(4) f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} = \frac{\cos x}{\sin x}$$

$$(5) f(x) = \sec x = \frac{1}{\cos x} = \frac{1}{\cos x}$$

$$(6) f(x) = \csc x = \frac{1}{\sin x} = \frac{1}{\sin x}$$

$$(1) \sin^2 a + \cos^2 a = 1$$

$$(2) \sin(2a) = 2 \sin a \cos a$$

$$(3) \cos(2a) = \cos^2 a - \sin^2 a$$

$$2 \cos^2 a - 1$$

$$1 - 2 \sin^2 a$$

$$\begin{aligned} \cos^2 a - \sin^2 a \\ 2 \cos^2 a - 1 \\ 1 - 2 \sin^2 a \end{aligned}$$

$$(4) \sec^2 a = 1 + \tan^2 a \quad \tan^2 a = \sec^2 a - 1$$

$$(5) \csc^2 a = 1 + \cot^2 a \quad \cot^2 a = \csc^2 a - 1$$

$$(6) \sin^2 a = \frac{1}{2} (1 - \cos(2a))$$

$$(7) \cos^2 a = \frac{1}{2} (1 + \cos(2a))$$

$$(8) \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$(9) \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$(10) \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$(11) \sin(a + 2\pi) = \sin a$$

$$(12) \cos(a + 2\pi) = \cos a$$

* NOTE

(1) $\cos x, \sec x$ even function

(2) $\sin x, \tan x, \cot x, \csc x$ odd function

صريح استحقاق الزاوية

* لكل طائفة خلق ابتداء -

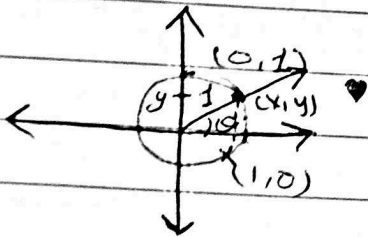
يكون على محور السينات الموجب

* الزاوية انا صشت عكس عقارب الساعة بنكون (+)

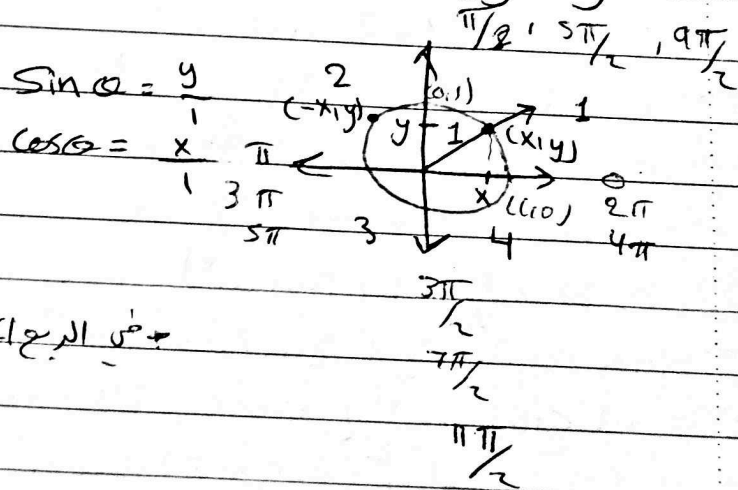
counter clockwise

عقارب الزاوية clockwise

✱ رائدة الوحدة ✱



دائرة مركزها $(0,0)$ وقطرها 1



• في الربع الأول كره (كل الأقارب) أحسنه موصية

 ~~$(\cos x, \sin x)$~~

$\sin x$	All
$\csc x$	
$\tan x$	$\cos x$
$\cot x$	$\sec x$

lecture 7

* دالة الاقترانات المثلثية *

X	Sinx	cosx	tanx	cotx	secx	cscx
360° or 2π	0	1	0	∞	1	∞
90° π/2	1	0	∞	0	∞	1
180° π	0	-1	0	∞	-1	∞
270° 3π/2	-1	0	∞	0	∞	-1
30° π/6	1/2	√3/2	1/√3	√3	2/√3	2
45° π/4	1/√2	1/√2	1	1	√2	√2
60° π/3	√3/2	1/2	√3	1/√3	2	2/√3

* اولى دالة المثلثات من دائرة الوحدة الباقية حذفا *

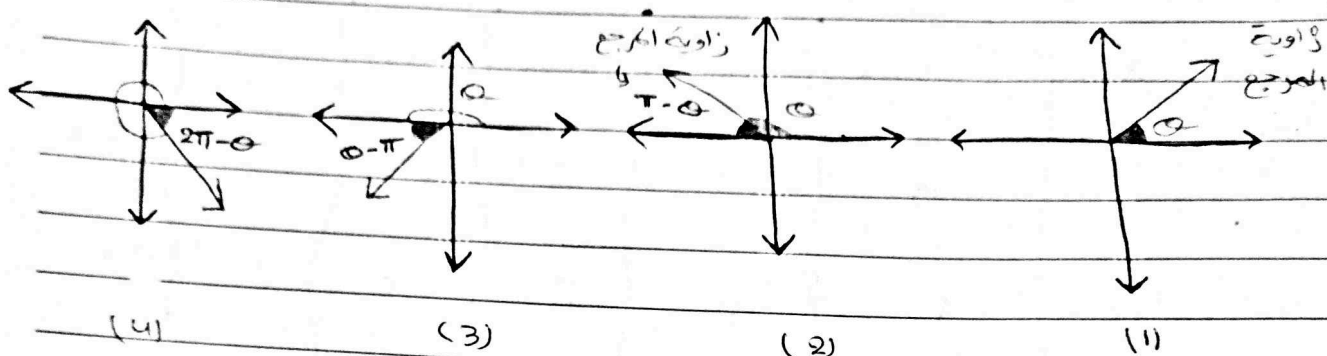
$$\tan \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1 \cdot 2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \quad \text{استخدام يعرفنا}$$

$$\csc \frac{\pi}{3} = \frac{1}{\sin \frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{1 \cdot 2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \quad (0,1)$$

$$\cot \frac{\pi}{4} = \frac{\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1 \quad (-1,0) \quad (1,0)$$

(0,-1)

زاوية المرجع : هي زاوية حادة محصورة بين محور السينات و ضلع اتجاه الكوة



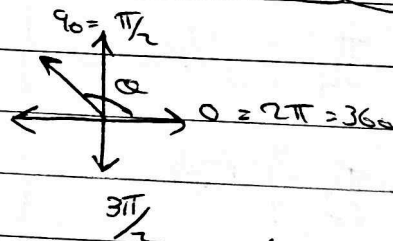
الزاوية محصورة بين ضلع الاتجاه الزاوية ومحور السينات (محصورة) حادة

التحويل

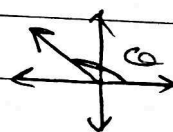
درجات $\rightarrow$ راديان $\leftarrow$ بضرب بـ $\frac{\pi}{180}$

راديان $\leftarrow$ درجات $\leftarrow$ بضرب بـ $\frac{180}{\pi}$

$$120^\circ \rightarrow 120^\circ \cdot \frac{\pi}{180} = \frac{2\pi}{3}$$



$$\frac{3\pi}{4} \rightarrow \frac{3\pi}{4} \cdot \frac{180}{\pi} = (3)(45) = 135^\circ$$



$$\text{or } 3 \left(\frac{\pi}{4} \right) \rightarrow 45^\circ = (3)(45) = 135^\circ$$

Examp:-

sin	cos
csc	sec
tan	cot
cot	tan

135°
 ① $\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{4\pi}{4} - \frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \#$

الربع الثاني (لـ)

330°
 ② $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{12\pi}{6} - \frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

الربع الرابع (-)

240°
 ③ $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\frac{4\pi}{3} - \pi\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

ربع ثالث (+)

H.w

① $\sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}$ Find $\sin(2\theta)$ $2\sin\theta\cos\theta$
 $2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} \#$

② Find $\sin(3x)\cos(5x)$ $\frac{1}{2}(\sin(3x-5x) + \sin(3x+5x))$
 $\frac{1}{2}(-\sin(2x) + \sin(8x)) = \frac{1}{2}(-0 + 0) = 0$

③ If $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \leq \theta < \pi$

Find:-



$\cos\theta = -\frac{\sqrt{3}}{2}$

$\csc\theta = 2$

$\sin^2\theta + \cos^2\theta = 1$ $1^2 + x^2 = 2^2$

$\sec\theta = -2/\sqrt{3}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$4 - 1 = x^2 = 3 \Rightarrow x^2$

$\tan\theta = -1/\sqrt{3}$

الربع الثاني (لـ)

$x = \sqrt{3}$

$\cot\theta = -\sqrt{3}$

④ Find

150°

csc	
sin	At
tan	cos
cot	sec

$$\textcircled{1} \sin\left(\frac{5\pi}{6}\right) \quad \sin\left(\frac{6\pi}{6} - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

135°

$$\textcircled{2} \sin\left(\frac{3\pi}{4}\right) \quad \sin\left(\frac{4\pi}{4} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

240°

$$\textcircled{3} \cot\left(\frac{4\pi}{3}\right) \quad \cot\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) = \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$$

225°

$$\textcircled{4} \csc\left(\frac{5\pi}{4}\right) = -\csc\left(\frac{5\pi}{4} - \frac{4\pi}{4}\right) = -\csc\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

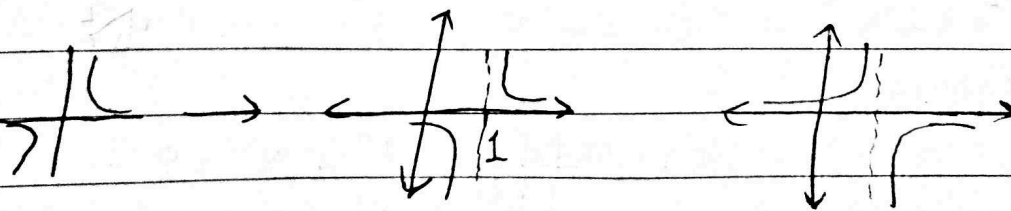
330°

$$\textcircled{5} \cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{12\pi}{6} - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\textcircled{6} \sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2} (1 - \cos \frac{2\pi}{12}) = \frac{1}{2} (1 - \cos \frac{\pi}{6}) =$$

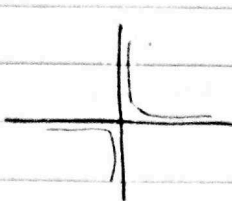
$$\frac{1}{2} \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2}{4} - \frac{\sqrt{3}}{4} = \frac{2 - \sqrt{3}}{4}$$

$$f(x) = \frac{1}{x-1}$$

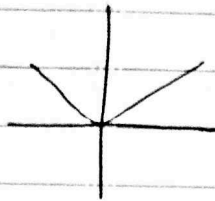


الدالة حول محور التوقع

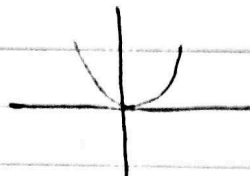
Lecture 8



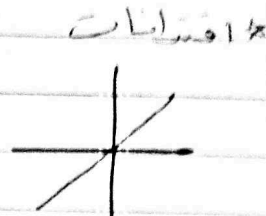
$$f(x) = \frac{1}{x}$$



$$f(x) = |x|$$



$$f(x) = x^2$$



$$f(x) = x$$

Domain: $\mathbb{R} \setminus \{0\}$

Range: $\mathbb{R} \setminus \{0\}$

(1)

Domain: $\mathbb{R}$

Range: $[0, \infty)$

(2)

Domain: $\mathbb{R}$

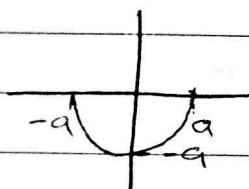
Range: $[0, \infty)$

(3)

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

(4)

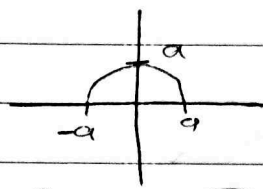


$$x^2 + y^2 = a^2 \rightarrow y = f(x) = -\sqrt{a^2 - x^2}$$

$$D_f = [-a, a]$$

$$R_f = [-a, 0]$$

(5)

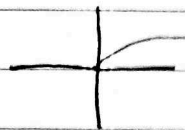


$$f(x) = \sqrt{a^2 - x^2}$$

$$D_f = [-a, a]$$

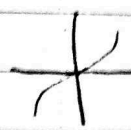
$$R_f = [0, a]$$

(6)



$$f(x) = \sqrt{x}$$

(7)



$$f(x) = x^3$$

(8)

تعريف

* Definition: - Suppose $c > 0$, $y = f(x)$

إذا حركنا الأعلى بمقدار (c) ، (نحرك بإيجابية)

① $y = f(x) + c$ ، Shift the graph $y = f(x)$ a distance (c) units upward

② $y = f(x) - c$ ، ~ ~ ~ ~ ~ downward

③ $y = f(x - c)$ ، ~ ~ ~ ~ ~ to the right

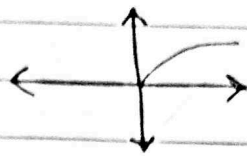
④ $y = f(x + c)$ ، ~ ~ ~ ~ ~ to the left

⑤ $y = -f(x)$ ، reflect ~ ~ ~ ~ ~ about the x-axis

⑥ $y = f(-x)$ ، reflect ~ ~ ~ ~ ~ y-axis

ق + 1 ايس سفير (5-0) ق + 2 ايس سفير (5-0)

Example :- $y = \sqrt{x}$



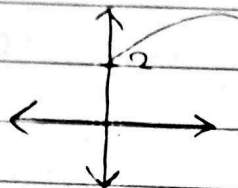
DF = $[0, \infty)$
RF = $[0, \infty)$

$y = \sqrt{x} - 1$



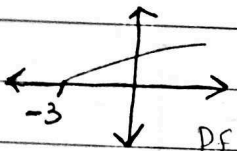
DF = $[0, \infty)$
RF = $[-1, \infty)$

$y = \sqrt{x} + 2$



DF = $[0, \infty)$
RF = $[2, \infty)$

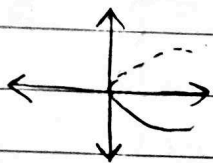
$y = \sqrt{x+3}$



DF = $[-3, \infty)$
RF = $[0, \infty)$

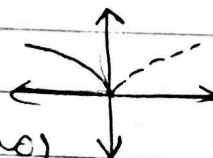
مع اول ج ليا و زحيا للبيلا

$y = -\sqrt{x}$



DF = $[0, \infty)$
RF = $(-\infty, 0]$

$y = \sqrt{-x}$



DF = $(-\infty, 0]$
RF = $[0, \infty)$

(ايس سفير (5-0) (ايس سفير (5-0)

Example :- find domain and range

اذا ايس سفير (5-0)

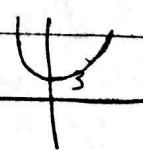
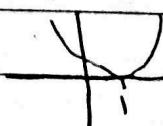
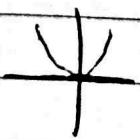
اذا ايس سفير (5-0)

(1) $f(x) = (x-1)^2 + 3$

$x^2 \rightarrow (x^2-1)^2 \rightarrow (x^2-1)^2 + 3$

DF = $\mathbb{R}$

RF = $[3, \infty)$



ايس سفير (5-0) ايس سفير (5-0) ايس سفير (5-0)

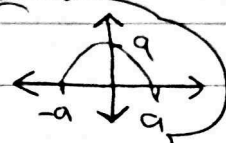
$$f(x) = (x-1)^2 + 3 = x^2 - 2x + 1 + 3 = x^2 - 2x + 4$$

$$D_f = \mathbb{R} \quad R_f = [f(1), \infty) = [3, \infty)$$

$$R = [R_{\min}, \infty)$$

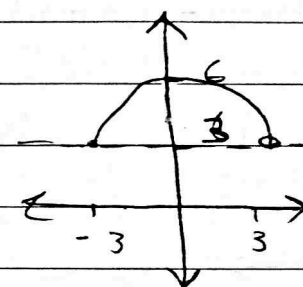
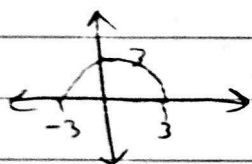
$$f(x) = \sqrt{9-x^2} + 3$$

$$f(x) = \sqrt{9-x^2} \quad D_f = [-3, 3] \quad R_f = [0, 3]$$



نقطه 3 و 0 را در نظر بگیر

$$\sqrt{9-x^2} \rightarrow \sqrt{9-x^2} + 3$$

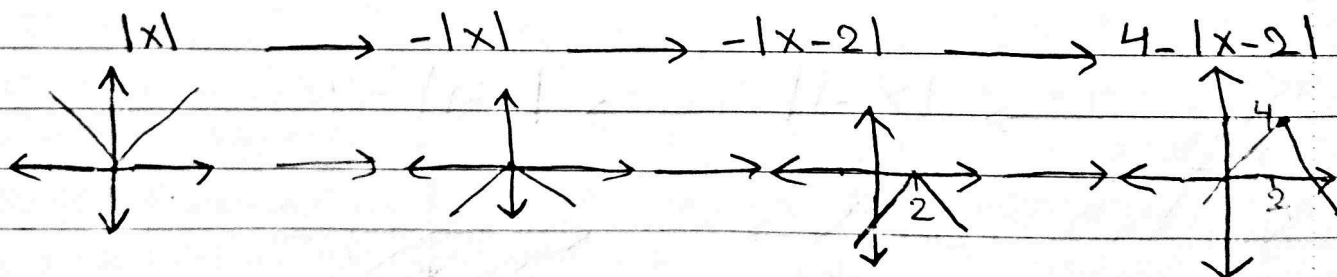


نقطه 3 و 6 را در نظر بگیر
مقادیرش را به دست آور

$$D = [-3, 3] \quad R = [3, 6]$$

(H.W) Find domain and range

$$f(x) = 4 - |x-2| = -|x-2| + 4$$

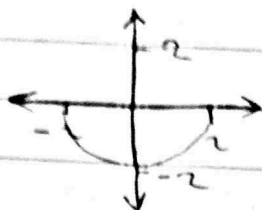


$$D_f = (-\infty, \infty)$$

$$R_f = [4, -\infty)$$

② $f(x) = -\sqrt{4-x^2}$

$f(x) = -\sqrt{4-x^2}$



DF: $[-2, 2]$
 RF: $[-2, 2]$

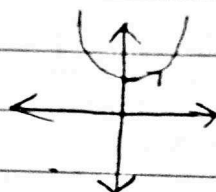
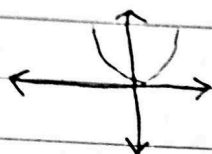
③ $f(x) = x^2 + 7$

x^2

$x^2 + 7$

DF: $\mathbb{R}$

RF: $[7, \infty)$



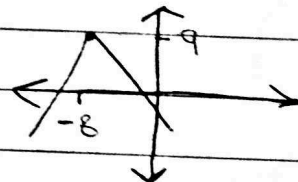
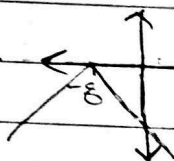
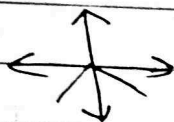
④ $f(x) = -|x+8| + 9$

$|x|$

$\rightarrow -|x|$

$\rightarrow -|x+8|$

$\rightarrow -|x+8| + 9$



DF: $\mathbb{R}$

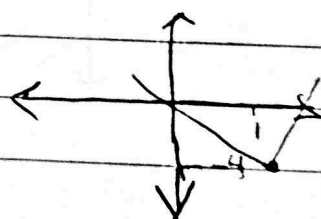
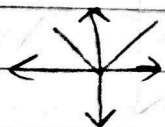
RF: $[-9, \infty)$

⑤ $f(x) = |x-1| - 4$

$|x|$

$\rightarrow |x-1|$

$\rightarrow |x-1| - 4$



DF: $\mathbb{R}$

RF: $[-4, \infty)$

Lecture 9

Definition: If f and g functions with domains D_f, D_g

① $(f \pm g)(x) = f(x) \pm g(x)$ with $D_{f \pm g} = D_f \cap D_g$ (تقاطع)
(المجموعة المكونة)

② $(f \cdot g)(x) = f(x) \cdot g(x)$ with $D_{f \cdot g} = D_f \cap D_g$ (تقاطع)

③ $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ with $D_{f/g} = D_f \cap D_g / \{x = g(x) = 0\}$ (تقاطع)
كل x يتخذ القيمتين

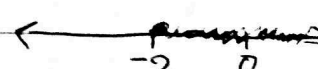
Example: $f(x) = \sqrt{x+2}$, $g(x) = \frac{1}{x}$

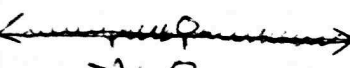
① $(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \frac{1}{x} = \frac{x\sqrt{x+2} - 1}{x}$

② $(f-g)(4) = f(4) - g(4) = \sqrt{4+2} - \frac{1}{4} = \frac{4\sqrt{6}-1}{4}$ (ملاحظة 1)

$(f-g)(4) = \frac{4\sqrt{4+2} - 1}{4} = \frac{4\sqrt{6}-1}{4}$

③ $D_{f-g} = D_f \cap D_g = [-2, \infty) \cap \mathbb{R} \setminus \{0\}$
 $[-2, 0) \cup (0, \infty) = [-2, \infty) \setminus \{0\}$

D_f 

D_g 

$f(x) = \sqrt{x+2}$
 $x+2 \geq 0$
 $x \geq -2$

$D_f = [-2, \infty)$

$x=0 \neq g(x) = \frac{1}{x} \leftarrow \mathbb{R}$

$D_g = \mathbb{R} \setminus \{0\}$

$\mathbb{R} \cap \mathbb{R} \setminus \{0\} = \mathbb{R} \setminus \{0\}$

* قاعدة: لما بدأنا المجال قبل ما نبنيها او نختار

Example: Find domain

$$\textcircled{1} f(x) = \frac{x}{x^2-9} \rightarrow \textcircled{1}$$

المقام
 $D_1 = \mathbb{R}$
 $D_2 = \mathbb{R}$

$$x^2-9=0, (x-3)(x+3), x=3, -3$$

$$Df = D_1 \cap D_2 / \{ \pm 3 \}$$

$$\mathbb{R} \cap \mathbb{R} / \{ \pm 3 \} = \mathbb{R} / \{ \pm 3 \}$$

$$\textcircled{2} f(x) = \frac{1}{\sqrt{1-x}} \rightarrow \textcircled{1}$$

$$* D_1 = \mathbb{R}$$

$$* D_2 = 1-x \geq 0 \rightarrow x \leq 1$$

$$(-\infty, 1]$$



$$(\sqrt{1-x})^2 = 0^2 = 0 \rightarrow 1-x=0 \rightarrow x=1$$

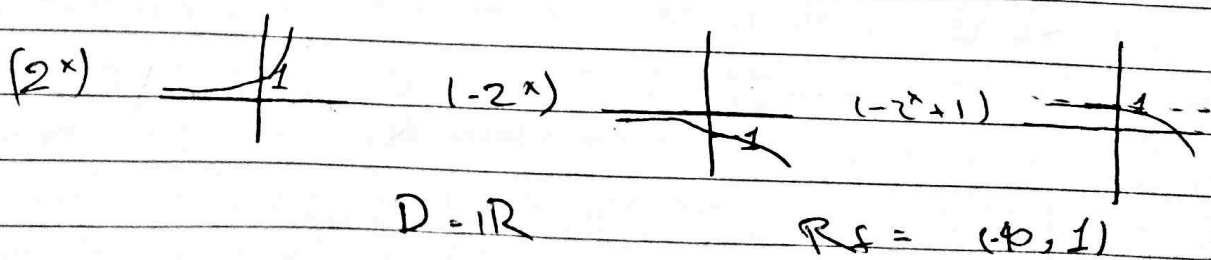
$$Df = D_1 \cap D_2 / \{ 1 \}$$

$$\mathbb{R} \cap (-\infty, 1] / \{ 1 \}$$

$$(-\infty, 1] / \{ 1 \}$$

$$(-\infty, 1)$$

Find the Range $f(x) = 1-2^x$



$$D = \mathbb{R}$$

$$Rf = (-\infty, 1)$$

Lesson 10

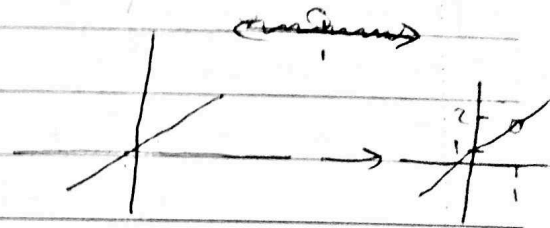
Example: find domain f

① $f(x) = \frac{x^2-1}{x-1} \rightarrow$ كثير حدود $\rightarrow \mathbb{R}$ اصفا، اتمام
 $x-1 \rightarrow$ كثير حدود $\rightarrow \mathbb{R}$ $x-1=0 \Rightarrow x=1$

$Df = \mathbb{R} \cap \mathbb{R} \setminus \{1\} = \mathbb{R} \setminus \{1\} = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$

$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)}$

حذف
الحد
المشترك



فاي x (x) فاي $x+1$ $(x+1)$
 $\frac{x^2-1}{x-1}$

② $f(x) = \sqrt{|x-5|-2}$ الجذر (زوجي ترسيم)

$|x-5|-2 \geq 0 \rightarrow |x-5| \geq 2$ $|x| \geq a \rightarrow x \geq a, x \leq -a$

$x-5 \geq 2$ or $x-5 \leq -2$ $Df = (-\infty, 3] \cup [7, \infty)$
 $x \geq 7$ or $x \leq 3$

لأننا 4 $\sqrt{-1}$ فهاي غلطا $[3, 7]$ القيم هاي بتجعل تحت الجذر سال

③ $f(x) = \sqrt{\frac{x}{x+1}} \neq \frac{\sqrt{x}}{\sqrt{x+1}}$ ③ $\frac{x}{x+1} \geq 0$

$Df = (-\infty, -1) \cup [0, \infty)$

لأنه لحد اتمام ناتج القسمة

① $x=0$

② $x+1=0, x=-1$

صنعت الكي تخلص الكي الكي

و و موجة

H.W

③ $f(x) = \frac{\sqrt{x}}{\sqrt{x+1}}$ $\sqrt{x} \rightarrow x=0 \rightarrow \leftarrow \frac{0}{-10}$

$\sqrt{x+1} \rightarrow x+1=0 \rightarrow x=-1 \rightarrow \leftarrow \frac{0}{-10}$

$\sqrt{x} \rightarrow D_f \rightarrow (0, \infty)$

$\sqrt{x+1} \rightarrow D_f \rightarrow [-1, \infty)$ $\boxed{x=-1}$ = نقطة

الاقتران = $(0, \infty)$

$D_f = (0, \infty) \cap [-1, \infty) \setminus \{-1\}$

④ $f(x) = \sqrt[3]{\frac{1}{x-1}}$ جذر تكعيبي

(عادي يمكن حله بـ 3 أو 6)

حاله هو مجال الاقتران في حقه

$\frac{1}{x-1} \rightarrow \mathbb{R}$ كمنصور

$x-1 \rightarrow \mathbb{R} \setminus \{1\}$ $\boxed{x=1}$ = نقطة

$D_{\frac{1}{x-1}} = \mathbb{R} \cap \mathbb{R} \setminus \{1\}$

$D_f = \mathbb{R} \setminus \{1\}$

lecture 11

Example: Find domain

$$D = \mathbb{R}$$

* خط مجال البسط ومجال المقام
* نستثنى المقامات المقام

$$(1) f(x) = \frac{2x+1}{3-|x|} \rightarrow \text{كثير حدود} \rightarrow B = \mathbb{R}$$

$$\text{مقام المقام} = 3 - |x| = 0 \rightarrow |x| = 3 \rightarrow x = 3, -3$$

$$D_f = D_{2x+1} \cap D_{3-|x|} \setminus [-3, 3] \rightarrow \mathbb{R} \cap \mathbb{R} \setminus [-3, 3] = \mathbb{R} \setminus [-3, 3]$$

$$(2) f(x) = \sqrt{x^2 - 5x + 6}$$

$$x^2 - 5x + 6 \geq 0$$

$$x^2 - 5x + 6 = 0 \quad (x-3)(x-2) \quad \text{أولاً خلال الصلة الترتيبية}$$

$$\begin{array}{c} (x-3) \quad \leftarrow \frac{1}{3} \rightarrow \\ (x-2) \quad \leftarrow \frac{1}{2} \rightarrow \end{array}$$

$$x = 3, 2$$

$$D_f = (-\infty, 2] \cup [3, \infty)$$

$$\begin{array}{c} \leftarrow + \quad - \quad + \rightarrow \\ \quad \quad \quad 2 \quad 3 \end{array}$$

$$(3) f(x) = \sqrt{x^2 - 5x + 6} + \frac{2x+1}{3-|x|}$$

$$D_f = D_2 \cap D_1$$

$$= ((-\infty, 2] \cup [3, \infty)) \cap (\mathbb{R} \setminus [-3, 3])$$

$$= (-\infty, -3) \cup (3, \infty) \cup (-3, 2]$$

$$\text{OR} =$$

$$(-\infty, 2] \cup (3, \infty) \setminus [-3, 3]$$

$$D_2 \leftarrow \text{---} \frac{1}{2} \text{---} \frac{1}{3} \text{---} \rightarrow$$

$$D_1 \leftarrow \text{---} \frac{1}{-3} \text{---} \frac{1}{3} \text{---} \rightarrow$$

$$D_1 \cup D_2 \leftarrow \text{---} \frac{1}{-3} \text{---} \frac{1}{2} \text{---} \frac{1}{3} \text{---} \rightarrow$$

$$D_1 \cup D_2 \leftarrow \text{---} \frac{1}{-3} \text{---} \frac{1}{2} \text{---} \frac{1}{3} \text{---} \rightarrow$$

lecture 12

Example : Find the domain and the range

① $f(x) = \sqrt{x-1} + 5$

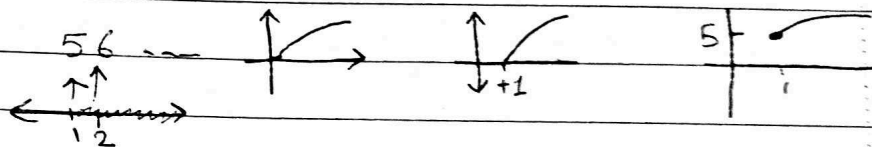
$x-1 \geq 0 \rightarrow x \geq 1$

$[1, \infty) \cap \mathbb{R}$

$Df = [1, \infty)$

$Rf = [5, \infty)$

$\sqrt{x} \quad \sqrt{x-1} \quad \sqrt{x-1} + 5$



② $f(x) = 3 \rightarrow \mathbb{R}$

$5 + \cos x \rightarrow \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

مجموع اقتران

القيمة، القيمة

$5 + \cos x = 0, \cos x \neq -5$

$-1 \leq \cos x \leq 1$

لا يوجد هنا، مقام

$Df = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

$\text{Range } f = [\frac{1}{2}, \frac{3}{4}]$

الحد الأدنى

$\frac{-1}{+5} \leq \cos x \leq \frac{1}{+5}$

$4 \leq 5 + \cos x \leq 6$

هنا نضرب في 5 بالحد الأدنى
أو نضرب في 5 بالحد الأعلى

$\frac{3}{4} \geq \frac{3}{5 + \cos x} \geq \frac{1}{2}$

$3 \left(\frac{1}{4} \geq \frac{1}{5 + \cos x} \geq \frac{1}{6} \right)$

$\frac{1}{2} \leq f(x) \leq \frac{3}{4}$

$$(3) f(x) = -3\sin^2 x$$

(sin) 2 saans
 $\sin$ ki value up to 1

$$Df = \mathbb{R} \quad \text{Range } f = [-3, 0]$$

substituting

$$-1 \leq \sin x \leq 1 \xrightarrow{-3} (0 \leq \sin^2 x \leq 1) \rightarrow -3 \leq f(x) \leq 0$$

sin substitution

$$-3\sin^2 x \text{ substitution}$$

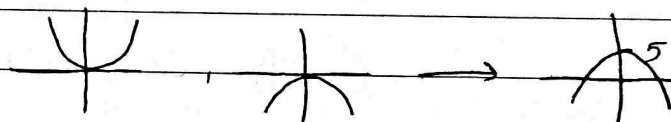
How

$$(4) f(x) = -x^2 + 5$$

$$x^2 \rightarrow -x^2 \rightarrow -x^2 + 5$$

$$Df = \mathbb{R}$$

$$Rf = [5, \infty)$$



$$(5) f(x) = \sqrt{4-x^2} + 3 \rightarrow \mathbb{R}$$



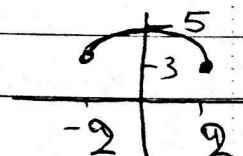
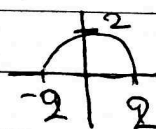
$$4-x^2 \geq 0 \rightarrow -2 \leq x \leq 2$$

$$\sqrt{4-x^2}$$

$$\sqrt{4-x^2} + 3$$

$$Df = [-2, 2]$$

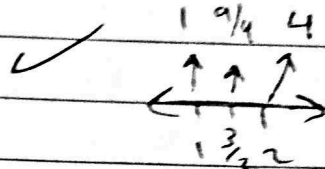
$$Rf = [3, 5]$$



$$\begin{aligned} \text{--- } f &\leftarrow \text{Domain} \\ \text{--- } g &\leftarrow \text{Range} \end{aligned}$$

Example: If Range of f is $[1, 2]$ Find the range of $g(x) = \frac{1}{1+(f(x))^2}$

$$1 \leq f(x) \leq 2$$

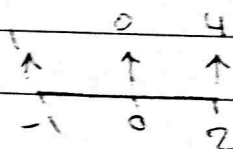


$$\frac{1}{1+(f(x))^2} \leq \frac{1}{1+1} \leq \frac{1}{1+4} \rightarrow 2 \leq 1+(f(x))^2 \leq 5$$

$$2 \leq (f(x))^2 + 1 \leq 5$$

$$\frac{1}{5} \leq g(x) \leq \frac{1}{2}$$

$$[1, 2]$$



من $[0, 4]$ إلى $(1, 4]$ من $[0, 4]$

نستعمل الترتيب لما يكون الباقى بالفترة

مثال

$$\text{Find domain } f(x) = \frac{1}{1+\sin(2x)} \rightarrow \mathbb{R}$$

$$1 + \sin(2x) = 0 \rightarrow \sin(2x) = -1$$

$$2x = \frac{3\pi}{2} + 2n\pi$$

$$x = \frac{3\pi}{4} + n\pi$$

$$Df = \mathbb{R} \setminus \left\{ \frac{3\pi}{4} + n\pi \right\}$$

Lecture 14

even
odd

زوج
فرد

Even and odd function

① $f(-x) = f(x)$ even function

متممات حول المحاور

② $f(-x) = -f(x)$ odd function

متممات حول نقطة الأصل

* Example - classify the functions as even, odd or neither

① $f(x) = x^2$
↓

$f(-x) = (-x)^2 = x^2 = f(x)$

even function

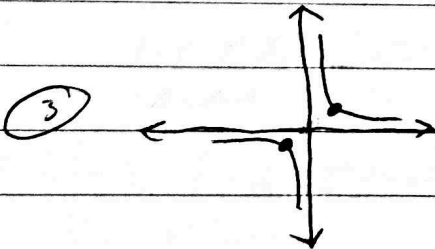
متممات حول المحاور

② $f(x) = \frac{x^5 - x}{1 + x^2}$
↓

$f(-x) = \frac{(-x)^5 - (-x)}{1 + (-x)^2} = \frac{-x^5 + x}{1 + x^2} = -\frac{x^5 - x}{1 + x^2}$

[odd function] $f(-x) = -f(x)$

بسالبة الاقتران الاول



odd function

متممات حول نقطة الأصل

④ $f(x) = x^5 + x$
 $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$
odd function

* Note: ① $\cos x$, $\sec x$ even function

$$\cos(-x) = \cos x \quad , \quad \sec(-x) = \sec x$$

② $\sin x$, $\csc x$, $\tan x$, $\cot x$ odd function

$$\sin(-x) = -\sin x$$

$$\csc(-x) = -\csc x$$

$$\tan(-x) = -\tan x$$

$$\cot(-x) = -\cot x$$

$$\textcircled{5} \quad f(x) = \frac{\tan x}{x^3 - x} \quad , \quad f(-x) = \frac{\tan(-x)}{(-x)^3 - (-x)}$$

$$= \frac{-\tan x}{-x^3 + x} = \frac{-\tan x}{-(x^3 - x)} = \frac{\tan x}{x^3 - x} = f(x)$$

even function

H.w

$$\textcircled{6} \quad f(x) = \frac{x^4 + 1}{\csc x} \quad , \quad f(-x) = \frac{(-x)^4 + 1}{-\csc x} = -\frac{x^4 + 1}{\csc x}$$

odd function

$\csc(-x)$

$$\textcircled{7} \quad f(x) = 2x - x^2 \quad , \quad f(-x) = -2x - (-x)^2 = -2x - x^2$$

$$f(x) = x(2 - x) \quad \rightarrow \quad f(-x) = -x(2 + x) = -(2x + x^2)$$

even or odd function

Lecture 15

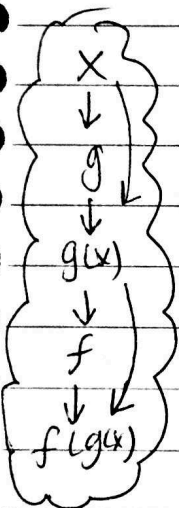
$$\sqrt{\square} = \frac{\sqrt{\square}}{\sqrt{\square}} \rightarrow \frac{D}{D} \cap$$

Composition of function : تركيب الاقترانات

- The composition of f with g $(f \circ g)(x) = f(g(x))$
(f circle g) or (f by g)
- The composition of g with f $(g \circ f)(x) = g(f(x))$

1
Domain $f \circ g = \{x \in D_g(x) \text{ and } g(x) \in D_f(x)\}$

$(f \circ g)(x)$ $\{x \in D_{\text{داخلي}} \text{ and } g(x) \in D_{\text{خارجي}}\}$



2
Domain $f \circ g = D_{\text{داخلي}} \cap D_{\text{الناتج عن التركيب حسب التسلسل}}$

Example :- $f(x) = \sqrt{x}$, $g(x) = \sqrt{2-x}$

Find each function and its domain?

نفس
المسألة

$$① (f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}}$$

$$((2-x)^{1/2})^{1/2} = (2-x)^{1/4} = \sqrt[4]{2-x}$$

Domain $f \circ g = \{x \in D_{\text{داخلي}} \text{ and } g(x) \in D_{\text{خارجي}}\}$
 $-g(x)$ $f(x)$

$$= \{x \in (-\infty, 2] \cap \sqrt{2-x} \in [0, \infty)\}$$

$$\{x \in (-\infty, 2] \text{ and } \sqrt{2-x} \geq 0\}$$

$$\{x \in (-\infty, 2] \text{ and } 2-x \geq 0\}$$

$$\{x \in (-\infty, 2] \text{ and } 1 - \frac{2}{2} \geq 0\}$$

$$(-\infty, 2] \cap (-\infty, 2] = (-\infty, 2]$$

$$f(x) = \sqrt{x}, x \geq 0$$

$$D_f = [0, \infty)$$

$$g(x) = \sqrt{2-x}$$

$$2-x \geq 0, 2 \geq x$$

$$D_g = (-\infty, 2]$$

② Domain $f \circ g = D_{\text{داخل}} \cap D$ الناتج من التركيب
 * الناتج من التركيب D
 بعد التبسيط

$$(-\infty, 2] \cap (-\infty, 2] = (-\infty, 2]$$

$$\sqrt{2-x}$$

$$2-x \geq 0$$

$$2 \geq x$$

$$\textcircled{2} (g \circ f)(x) = g(f(x))$$

$$(-\infty, 2]$$

$$g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$$

$$\Rightarrow \sqrt{2-\sqrt{x}}$$

$$2-\sqrt{x} \geq 0$$

$$2 \geq \sqrt{x}$$

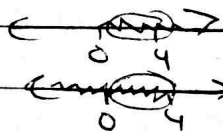
$$4 \geq x$$

$$D = (-\infty, 4]$$

الناتج
من
التركيب

$$* D_{g \circ f} = D_{\text{داخل}} \cap D$$

$$[0, \infty) \cap (-\infty, 4] = [0, 4]$$



$$* D_{g \circ f} = \{ x \in D_{\text{داخل}} \text{ and } f(x) \in D \}$$

$$= [0, 4]$$

$$\text{Find } (g \circ f)(a) =$$

نأخذ $a=9$ من ضمن المجال للاختبار لأن a أي رقم اعرفه
 (يجب أن تكون من المجال)

$$g(f(9)) = g(3) = \sqrt{2-3} = \text{undefined}$$

$$9 \notin [0, 4]$$

ولا حاجة لاختبار باقي الأرقام لأن 9 ليس من المجال

نأخذ بالاول من المجال

$$③ (f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

$[0, \infty)$

$$D_{f \circ f} = [0, \infty) \cap [0, \infty) = [0, \infty)$$

$$[0, \infty) = \sqrt{x} \text{ JLS}$$

$$[0, \infty) = \sqrt[4]{x} \text{ JLS}$$

$$④ (g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

$[-2, 2]$

$$D_{g \circ g} = (-\infty, 2] \cap [-2, \infty)$$

$$[-2, 2]$$

$$(-\infty, 2] \leftarrow \sqrt{2-x} \text{ JLS}$$

$$\leftarrow \sqrt{2-\sqrt{2-x}} \text{ JLS}$$

$$2 - \sqrt{2-x} \geq 0$$

$$2 \geq \sqrt{2-x}$$

$$4 \geq 2-x$$

$$-4 \leq -2+x$$

$$-2 \leq x$$

$$[-2, \infty)$$

* f odd, g odd

$$*(F+g)(x) = (F+g)(-x) = f(-x) + g(-x)$$

$$= -f(x) + -g(x) \rightarrow -(f(x) + g(x)) =$$

$$F+g = \text{odd}$$

$$*(Fg)(x) = (Fg)(-x) = F(-x) \cdot g(-x) = +F(x) \cdot g(x)$$

$$Fg = \text{even}$$

$$*(F-g)(x) = F(x) - g(x) = -F(x) + g(x) = -(f(x) - g(x))$$

$$F-g = \text{odd}$$

$$*\left(\frac{f}{g}\right)(x) = \frac{-f(x)}{-g(x)} = +\frac{f(x)}{g(x)}$$

$$\frac{f}{g} = \text{even}$$

* F odd, g even

$$*(F+g)(x) = F(-x) + g(-x) = -F(x) + g(x)$$

$$*(f \cdot g)(x) = -F(x) \cdot g(x) = \text{odd}$$

$$(f/g)(x)$$

lecture 16

Example:- $f(x) = \frac{3}{x}$, $g(x) = \frac{5x}{x-2}$ Find $(g \circ f)(x)$ and Domain $(g \circ f)(x)$?

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3}{x}\right) = \frac{5 \cdot \frac{3}{x}}{\frac{3}{x} - 2} = \frac{\frac{15}{x}}{\frac{3-2x}{x}}$$

$$\frac{15}{x} = \frac{x}{3-2x} = \frac{15}{3-2x} \rightarrow \text{قبل بسطها}$$

$$f(x) = \frac{3}{x} \rightarrow D_f = \mathbb{R} \setminus \{0\}$$

كل $x \neq 0$

$$g(x) = \frac{5x}{x-2} \rightarrow \mathbb{R} \quad x-2=0 \rightarrow x=2 \rightarrow D_g = \mathbb{R} \setminus \{2\}$$

Domain $g \circ f$ = الناتج من التركيب $D_f \cap D_g$ له التقييد

$$D_{\frac{3}{x}} \cap D_{\frac{15}{3-2x}} \rightarrow \mathbb{R} \setminus \{0\} \cap \mathbb{R} \setminus \left\{\frac{3}{2}\right\} = \mathbb{R} \setminus \left\{0, \frac{3}{2}\right\}$$

$$\frac{15}{3-2x} \rightarrow \mathbb{R}$$

$$3-2x=0$$

$$\frac{3}{2} = \frac{2x}{2}$$

$$x = \frac{3}{2}$$

$$D_{\frac{15}{3-2x}} = \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$$

Example:- Find domain $f(x) = \sqrt{3-x}$

ناتج من تركيب اقتصر

$$g(x) = \sqrt{x}, h(x) = \sqrt{3-x} \rightarrow f(x) = (h \circ g)(x) = h(g(x)) = \sqrt{3-\sqrt{x}}$$

$$D_{h \circ g} = D_{\sqrt{x}} \cap D_{\sqrt{3-x}} = [0, \infty) \cap (-\infty, 3] = [0, 3]$$

$$= \sqrt{3-\sqrt{x}}$$

$$\sqrt{x} \rightarrow x \geq 0 \rightarrow [0, \infty)$$

$$\sqrt{3-x} \rightarrow 3-x \geq 0 \rightarrow 3 \geq x \rightarrow x \leq 3 \rightarrow (-\infty, 3]$$

lecture 17

Example:- $(f \circ g)(x) = 6x^2 - 10x + 5$, $f(x) = 2x + 1$ Find $g(x)$?

$$(f \circ g)(x) = f(g(x)) = 2g(x) + 1 = 6x^2 - 10x + 5$$

$$\frac{2g(x)}{2} = \frac{6x^2 - 10x + 4}{2} \rightarrow g(x) = 3x^2 - 5x + 2$$

$$g(0) = 0 + 0 + 2 = 2$$

Example $(f \circ g)(x) = x^2 + 5$, $g(x) = 2x + 3$ Find $f(x)$?

$$f(g(x)) = f(2x+3) = x^2 + 5 \rightarrow \text{let } y = 2x + 3$$

$$\frac{y-3}{2} = \frac{2}{2}x \rightarrow x = \frac{y-3}{2}$$

$$f(y) = \left(\frac{y-3}{2}\right)^2 + 5, f(z) = \left(\frac{z-3}{2}\right)^2 + 5, f(m) = \left(\frac{m-3}{2}\right)^2 + 5$$

$$f(x) = \left(\frac{x-3}{2}\right)^2 + 5$$

Example:- If domain $f = [1, 3]$ and

$g(x) = -2x + 1$ Find domain $f \circ g$?

$$\text{Domain } f \circ g = \left[x \in D \text{ and } g(x) \in D_{f, \mathbb{R}} \right]$$

$$\left[x \in \mathbb{R} \text{ and } -2x + 1 \in [1, 3] \right] \rightarrow \left[x \in \mathbb{R} \text{ and } \frac{0}{-2} \leq \frac{-2x + 1}{-2} \leq \frac{2}{-2} \right]$$

$$\left[x \in \mathbb{R} \text{ and } 0 \geq x \geq -1 \right] \rightarrow \mathbb{R} \cap [-1, 0] = [-1, 0]$$

نفس الشيء

$$\begin{aligned} f(x) &= x^2 \\ \downarrow \\ f(z) &= z^2 \\ \downarrow \\ f(y) &= y^2 \end{aligned}$$

نفس الشيء

H.W

① $f(x) = x^2 + 3$, $g(x) = \sqrt{3-x}$ Find $(f \circ g)(x)$, $D(f \circ g)$
 $f(g(2))$, $(f \circ g)(4)$, $(g \circ g)(x)$, $Dg \circ g$, $g \circ f$, $Dg \circ f$

① $f \circ g(x) = (\sqrt{3-x})^2 + 3 = f \circ g(x) = 3 - x + 3 = 6 - x$

② $D f \circ g = g(x) = \sqrt{3-x}$ $3-x \geq 0 \rightarrow 3 \geq x$ $[-\infty, 3]$
 $f \circ g(x) = 6 - x$ $\mathbb{R}$
 $[-\infty, 3] \cap \mathbb{R} \rightarrow [-\infty, 3]$

③ $f(g(2)) = 6 - 2 = 4$

④ $f(g(4)) =$ ليس له معنى $\sqrt{3-4} = \sqrt{-1}$ undefined

⑤ $g(g(x)) = \sqrt{3 - \sqrt{3-x}}$

⑥ $Dg \circ g = 3 - \sqrt{3-x} \geq 0 \rightarrow 3 \geq \sqrt{3-x} \rightarrow \frac{9}{3} \geq 3-x \rightarrow 6 \geq -x$
 $-6 \leq x$ $[-6, \infty)$, $Dg(x) = [-\infty, 3] \rightarrow [-6, \infty) \cap (-\infty, 3] = [-6, 3]$

⑦ $g \circ f = g(f(x)) = \sqrt{3 - (x^2 + 3)} = \sqrt{3 - x^2 - 3} = \sqrt{-x^2}$
 $|x|$

⑧ $Dg \circ f = Df(x) = \mathbb{R}$ $Dg \circ f = \mathbb{R}$

② $(f \circ g)(x) = 3x^2 + 3x + 2$, $f(x) = 3x + 5$ Find $g(x)$?

$f(g(x)) = 3g(x) + 5 = 3x^2 + 3x + 2 \rightarrow \frac{3g(x)}{3} = \frac{3x^2 + 3x - 3}{3}$

$g(x) = x^2 + x - 1$

③ $(f \circ g)(x) = x^2 + 6x + 6$, $g(x) = x + 1$ Find $f(x)$?

$f(g(x)) = f(x+1) = x^2 + 6x + 6$ let $y = x+1 \rightarrow x = y-1$

$f(y) = (y-1)^2 + 6(y-1) + 6 \rightarrow f(x) = (x-1)^2 + 6(x-1) + 6$
 $= f(x) = x^2 - 2 + 1 + 6x - 6 + 6 = f(x) = x^2 + 6x - 1$

④ $f(x) = \frac{x}{x+1}$, $g(x) = x^3$, $h(x) = x+2$

* $(f \circ g \circ h)(x) = f(g(h(x))) = g(h(x)) = (x+2)^3$

$f(g(h(x))) = \frac{(x+2)^3}{(x+2)^3 + 1}$ قانون السكيب $(x+y)^3 = (x^3) + (3x^2y) + (3xy^2) + (y^3)$

* $(h \circ g)(x) = h(g(x)) = x^3 + 2$

* $(f \circ g)(x) = f(g(x)) = \frac{x^3}{x^3 + 1}$

* $f \circ f(x) = \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} \rightarrow \frac{\frac{x}{x+1}}{\frac{x}{x+1} + \frac{x+1}{x+1}} \rightarrow \frac{\frac{x}{x+1}}{\frac{x + x^2 + 1}{x+1}} \rightarrow \frac{x}{x^2 + x + 1}$

If g even $f \circ g \rightarrow f(g(-x)) = f(g(x))$ even
 $\sim \sim$ odd $f \circ g \rightarrow f(g(-x)) = f(-g(x))$ odd

اقتران واحد لواحد؟

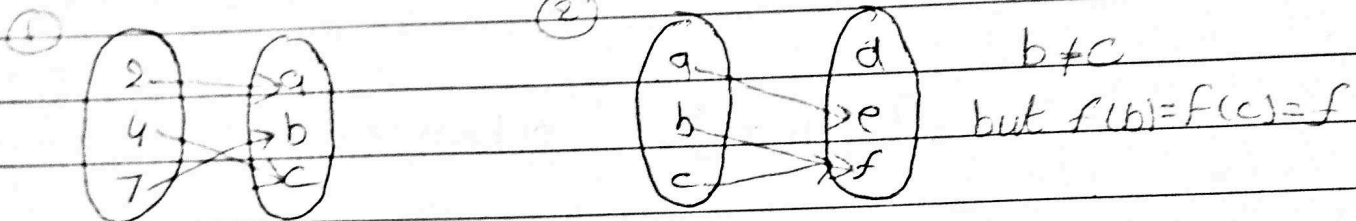
One to one function

f is called one to one function if $f(x_1) \neq f(x_2)$

whenever $x_1 \neq x_2$

* إذا كان لدينا دالة f من مجموعة A إلى مجموعة B ، فإن f تكون دالة واحدة لواحد إذا كان لكل عنصر في A صورة واحدة في B ، والعكس صحيح.

Example :-



one to one function

not one to one

كل دالة واحدة لواحد هي دالة

Remark = Test for one to one function : If $f(a) = f(b) \rightarrow a = b$ then f is one to one function

أي دالة تحقق الشرط أعلاه هي دالة واحدة لواحد.

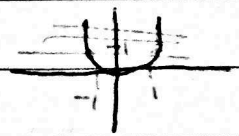
Example:- Is $f(x) = x^2$ one to one function?

Not one to one function

$-1 \neq 1$ but $f(-1) = f(1) = 1$

$-2 \neq 2$ but $f(-2) = f(2) = 4$

not one to one



دالة واحدة لواحد

$$f(a) = f(b)$$

$$\sqrt{a^2} = \sqrt{b^2}$$

$$|a| = |b| \Rightarrow a = b \text{ or } a = -b$$

not one to one

Example:- Is $f(x) = 3x - 2$ one to one function?

$$\text{let } f(a) = f(b)$$

$$3a - 2 = 3b - 2$$

$$3a = 3b - 2 + 2$$

$$\frac{3a}{3} = \frac{3b}{3} \Rightarrow a = b$$

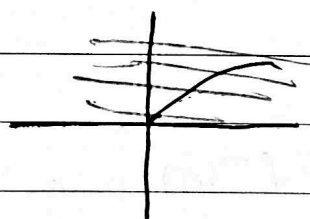
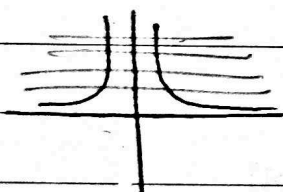
$\therefore$ One to one function

* Horizontal line test :- اختبار الخط الأفقي

→ الدالة هي one to one إذا لم تقطع الخط الأفقي

→ الخط الأفقي one to one إذا لم تقطع الخط الأفقي

ولا تقطع إلا نقطة واحدة إذا تقطعت بأكثر من نقطة يكون Not one to one



Not one to one

one to one

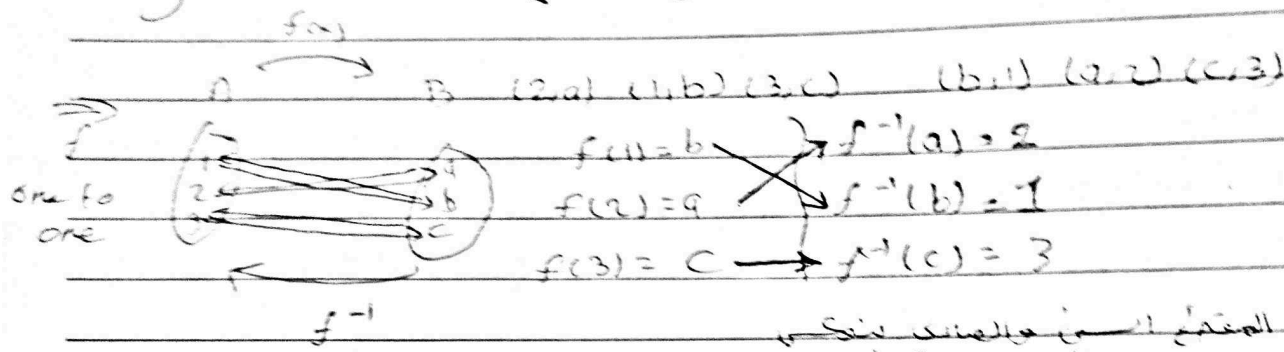
فقط نقطة واحدة تقطع الخط الأفقي

lecture 19

Inverse function

Theorem: f has an inverse f^{-1} if and only if f is one to one function.

Definition: let f is one to one, then its inverse is defined by $f^{-1}(y) = x \iff y = f(x)$



$$Df^{-1} = Rf$$

$$f(2) = a$$

$$Rf^{-1} = Df$$

$$f^{-1}(f(2)) = f^{-1}(a) = 2 = f^{-1}(a)$$

$$f^{-1}(c) = 3$$

$$f(f^{-1}(c)) = f(3) = c = f(3)$$

Remark:

$$① \text{ Domain } f^{-1} = \text{Range } f = B$$

$$② \text{ Range } f^{-1} = \text{Domain } f = A$$

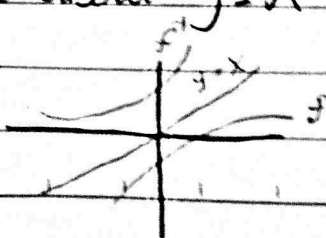
$$③ (f \circ f^{-1})(x) = x = f(f^{-1}(x)) = x \quad \forall x \in B$$

$$④ (f^{-1} \circ f)(x) = x, \quad \forall x \in A$$

$$⑤ f \text{ and } f^{-1} \text{ Symmetric about } y = x$$

المنعكس (المرآة)

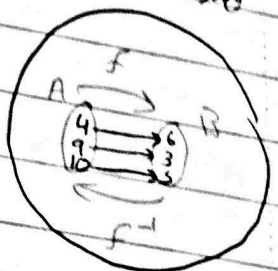
(y=x) حول



for all $x \in I$ there exist

inverse $f(4)=6, f(9)=3$

ما الخطم اجل كل المايب



$$\textcircled{1} f^{-1}(6) = 4$$

$$f^{-1}(f(4)) = f^{-1}(6) \rightarrow 4 = f^{-1}(6)$$

$$f^{-1}(f(x)) = x \quad \forall x \in D_f$$

$$\textcircled{2} f(10) = 5$$

$$f^{-1}(5) = 10 \rightarrow f(f^{-1}(5)) = f(10) \rightarrow 5 = f(10)$$

$$\textcircled{3} f^{-1}(3) = 9$$

$$f^{-1}(f(9)) = f^{-1}(3) \rightarrow f^{-1}(3) = 9$$

Lecture 20

Example:- Find $f^{-1}(x)$ for

$$\textcircled{1} f(x) = \sqrt{3x-2} \rightarrow \textcircled{1} y = \sqrt{3x-2} \rightarrow y^2 = 3x-2$$

$$y^2 + 2 = 3x \quad \Rightarrow \quad x = \frac{y^2 + 2}{3} = f^{-1}(x) = \frac{x^2 + 2}{3}$$

الخطوات لفضل الاقتران f^{-1} ① نبدأ $f(x)$ و y ② موضوع للقائمية
نبدأ $x \leftarrow f^{-1}(y)$ و $x \leftarrow y$

② $f(x) = \frac{x+1}{x-1} \rightarrow \frac{y}{1} = \frac{x+1}{x-1} = y(x-1) = x+1$

~~$$yx - y = x + 1 \rightarrow yx - x = y + 1 \rightarrow x(y - 1) = y + 1$$~~

$$X = \frac{y+1}{y-1} \rightarrow f^{-1}(x) = \frac{x+1}{x-1}$$

Find Domain f^{-1}

$$f^{-1}(x) = \frac{1+x}{x-1} \rightarrow \mathbb{R} \quad x-1=0 \quad Df^{-1} = \mathbb{R} \cap \mathbb{R} \setminus \{1\}$$

$$x-1 \rightarrow \mathbb{R} \quad \boxed{x=1} \quad = \mathbb{R} \setminus \{1\}$$

$$Df^{-1} = \text{Range } f = \mathbb{R} \setminus \{1\}$$

* Find Range f^{-1} = Domain $f = \mathbb{R} \setminus \{1\}$

اقتزان (one to one) ←

Example :- Find Domain and Range for $f(x) = \frac{x+2}{2x-1}$?

$$\text{Domain } f = \mathbb{R} \cap \mathbb{R} \setminus \left[\frac{1}{2} \right]$$

$$2x-1=0 \quad x=\frac{1}{2}$$

$$\text{Range } f = \text{Domain } f^{-1} = \mathbb{R} \setminus \left[\frac{1}{2} \right]$$

$$\text{Find } f^{-1} = y = \frac{x+2}{2x-1} \Rightarrow 2xy - y = x+2 \rightarrow 2xy - x = y+2$$

$$x(2y-1) = y+2 \rightarrow x = \frac{y+2}{2y-1} \rightarrow f^{-1}(x) = \frac{x+2}{2x-1}$$

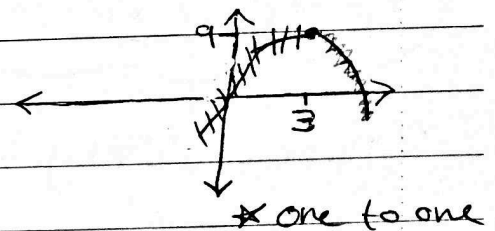
Lecture :- $f(x) = 6x - x^2$, $x \geq 3$ Find $f^{-1}(x)$?

21

اقتزان الترتيب من one to one function من $x \geq 3$

$$y = 6x - x^2$$

عشان نعمل x موضوع
للقانون نعمل الاحمال مربع



$$-y = -6x + x^2$$

$$-y = x^2 - 6x + 9 - 9 \quad \left(\frac{1}{2} \times -6\right)^2 = 9$$

$$-y = (x-3)(x-3) - 9$$

$$\sqrt{-y+9} = \sqrt{(x-3)^2}$$

١) عشان يكون معامل x^2 موجب

٢) نصف فطرح $\left(\frac{1}{2} \times -6\right)^2 = 9$

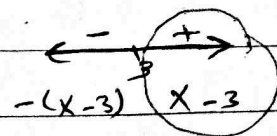
٣) نحل

$$\sqrt{9-y} = |x-3| \rightarrow$$

نضيق

$$x-3=0 \rightarrow x=3$$

$$\sqrt{9-y} = x-3 \rightarrow x = \sqrt{9-y} + 3$$



$$f^{-1}(x) = \sqrt{9-x} + 3$$

$$f \circ f^{-1}(x) = x \quad \checkmark$$

عشان نتحقق

$$f^{-1} \circ f(x) = x \quad \checkmark$$

انه f^{-1} لل f من $x \geq 3$ يتحقق الشرط

Example 1 - Determine whether f and g are inverse functions

$$f(x) = x^3 + 3x^2 + 3x + 1 = (x+1)^3, \quad g(x) = x^{1/3} - 1$$

$$(f \circ g)(x) \stackrel{?}{=} x \quad (f \circ g)(x) = f(g(x)) = f(x^{1/3} - 1)$$

$$(g \circ f)(x) \stackrel{?}{=} x \quad (x^{1/3} - 1 + 1)^3 = x \quad \checkmark$$

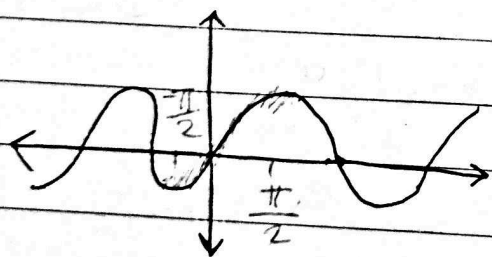
$$g \circ f(x) = g(f(x)) = ((x+1)^3)^{1/3} - 1 = x+1-1 = x \quad \checkmark$$

f, g are inverse $f = g^{-1} \quad g = f^{-1} \quad \checkmark$

Lecture 22

Inverse Trigonometric Function الاقترانات العكسية للأقترانات المثلثية

①

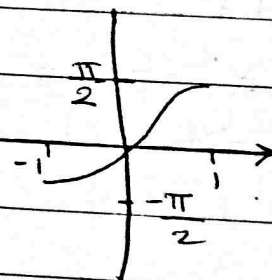


$$f(x) = \sin x$$

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$D \sin x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$R \sin x = [-1, 1]$$



$$f(x) = \sin^{-1} x$$

$$x \in [-1, 1]$$

$$\sin^{-1} x = \arcsin x$$

$$D \sin^{-1} x = [-1, 1]$$

$$R \sin^{-1} x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

* Remarks-

- ① $D \sin x = [-1, 1]$
- ② $R \sin x = [-\frac{\pi}{2}, \frac{\pi}{2}]$
- ③ $\sin(\sin^{-1}x) = x \quad \forall x \in [-1, 1]$
- ④ $\sin^{-1}(\sin x) = x \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- ⑤ $\sin^{-1}(-x) = -\sin^{-1}x$ ($\sin^{-1}x$ odd function)

بالا افتراضا الوحي

$$\sin^{-1}x \neq \frac{1}{\sin x}$$

$$f^{-1} \neq \frac{1}{f}$$

* Example:- Find

هناك محال ثابت

$$\textcircled{1} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

لا يمكن من حيث الـ $\sin$

لا على صورة $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\textcircled{2} \sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\textcircled{3} \sin^{-1}\left(\sin \frac{\pi}{7}\right) = \text{في الفترة} = \frac{\pi}{7}$$

$$\sin^{-1}(\sin x) = x$$

$$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\textcircled{4} \sin^{-1}\left(\sin 11\frac{\pi}{6}\right) = \text{ليست في الفترة}$$

$$-\frac{\pi}{2} \leq \frac{\pi}{7} \leq \frac{\pi}{2}$$

330° ربع، ربع، ربع

$$(-)\sin \quad \sin^{-1}\left(-\sin\left(\frac{12\pi}{6} - \frac{11\pi}{6}\right)\right)$$

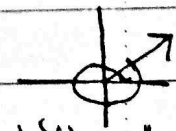
$$\sin^{-1}\left(-\sin\left(\frac{\pi}{6}\right)\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$$

↓ or

$$-\sin^{-1}\left(\sin \frac{\pi}{6}\right) = -\frac{\pi}{6}$$

ليست في الفترة $60 \times 7 = 420$

$$\textcircled{5} \sin^{-1}\left(\sin 7\frac{\pi}{3}\right) = \sin^{-1}\left(+\sin\left(\frac{7\pi}{3} - 2\pi\right)\right)$$

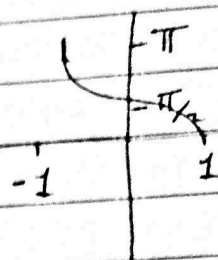
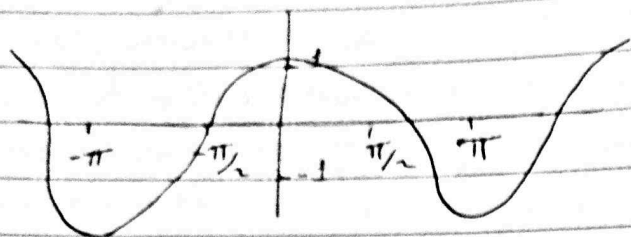


في الربع الأول

$$\sin^{-1}\left(\sin \frac{\pi}{3}\right) = +\frac{\pi}{3} \neq$$

نخرج منها -2π

lecture 23



$$f(x) = \cos x, x \in [0, \pi]$$

$$f(x) = \cos^{-1} x, x \in [-1, 1]$$

نأخذ هذا الجزء فقط عشان يكون (فترة) (1-1)

are cosine

المضي $[0, \pi]$

* Remark :-

$$\textcircled{1} D_{\cos^{-1}} = [-1, 1]$$

$$\textcircled{2} R_{\cos^{-1}} = [0, \pi]$$

$$\textcircled{3} \cos(\cos^{-1}(x)) = x, \forall x \in [-1, 1]$$

$$\textcircled{4} \cos^{-1}(\cos x) = x, \forall x \in [0, \pi]$$

$$\textcircled{5} \cos^{-1}(-x) = \pi - \cos^{-1}x \quad (\cos^{-1}x \text{ not even (not odd)})$$

Example : Find

قيمة العكس

قيمة العكس

$$\textcircled{1} \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\textcircled{2} \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4}$$

أكبر من 1 (البعض اربع)

$$\textcircled{3} \cos^{-1}\left(\cos \frac{11\pi}{6}\right) = \cos^{-1}(\cos x) = x \quad \forall x \in [0, \pi]$$

$$\cos^{-1}\left(\cos \left(2\pi - \frac{11\pi}{6}\right)\right) = \cos^{-1}\left(\cos \frac{\pi}{6}\right) = \frac{\pi}{6}$$

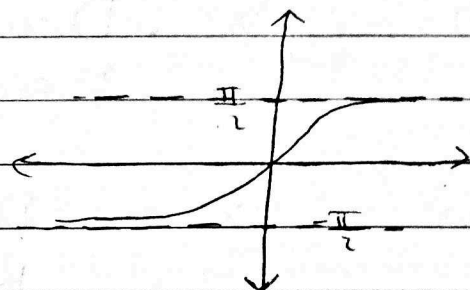
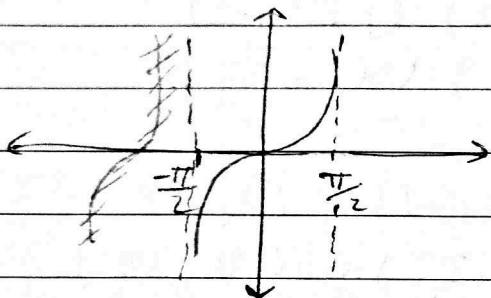
$$[-1, 1] \cos$$

مسألة 24

$$(4) \cos^{-1}(\cos \frac{7\pi}{6}) = \cos^{-1}(-\cos(\frac{7\pi}{6} - \pi)) = \cos^{-1}(-\cos(\frac{\pi}{6}))$$

$$\pi - \cos^{-1}(\cos(\frac{\pi}{6})) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Lecture 24



$$f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$f(x) = \tan^{-1} x, x \in \mathbb{R}$$

arc tan x

Remarks :-

- ① $D_{\tan^{-1}x} = \mathbb{R}$
- ② $R_{\tan^{-1}x} = (-\frac{\pi}{2}, \frac{\pi}{2})$
- ③ $\tan(\tan^{-1}x) = x \quad \forall x \in \mathbb{R}$
- ④ $\tan^{-1}(\tan x) = x \quad \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- ⑤ $\tan^{-1}(-x) = -\tan^{-1}x$ ($\tan^{-1}x$ odd function)

Example : Find

$$\textcircled{1} \tan^{-1}(1) = \frac{\pi}{4} \quad \text{حيث الفترة (قابل)}$$

$$\textcircled{2} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \quad \text{حيث الفترة (قابل)}$$

$$\textcircled{3} \tan^{-1}\left(\tan \frac{\pi}{6}\right) = \frac{\pi}{6} \quad \tan^{-1}(\tan x) = x \quad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\ast f(x) = \csc^{-1}x, \quad D_{\csc^{-1}x} = (-\infty, -1] \cup [1, \infty)$$

$$R_{\csc^{-1}x} = (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$$

$$\ast f(x) = \sec^{-1}x, \quad D_{\sec^{-1}x} = (-\infty, -1] \cup [1, \infty)$$

$$R_{\sec^{-1}x} = [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$$

$$\ast f(x) = \cot^{-1}x, \quad D_{\cot^{-1}x} = \mathbb{R}$$

$$R_{\cot^{-1}x} = (0, \pi)$$

$$\tan^{-1}x = \frac{1}{\tan x}, \quad \cot^{-1}x = \frac{1}{\tan^{-1}x}$$

* Example :- find

$$\textcircled{1} \sec^{-1}(2) = \frac{\pi}{3} \rightarrow \frac{1}{\cos x} = 2 \rightarrow \cos x = \frac{1}{2} \quad \text{حيث الفترة جوب}$$

مقبول الى $\frac{\pi}{3}$ لانه مقبل بؤيفة وياضه يقع بين فترة الفترة

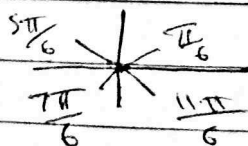
$$\textcircled{2} \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

$$\frac{\cos x}{\sin x} = \frac{\sqrt{3}}{\frac{1}{2}}$$

$$\textcircled{3} \cot^{-1}(\sqrt{3}) = -\cot(\sqrt{3}) = -\frac{\pi}{6} \quad \text{ليست حيث الفترة}$$

$$\frac{\cos x}{\sin x} = \frac{\sqrt{3}}{\frac{1}{2}}$$

$$(0, \pi) \leftarrow \frac{5\pi}{6}$$



Lecture 25

Example :- Find domain $f(x) = \sin^{-1}(3x+1)$?

$$D \sin^{-1} = [-1, 1]$$

$$\underset{-1}{-1} \leq \underset{-1}{3x+1} \leq \underset{-1}{1} \rightarrow -2 \leq 3x \leq 0$$

$$-\frac{2}{3} \leq x \leq 0, D f(x) = \left[-\frac{2}{3}, 0\right]$$

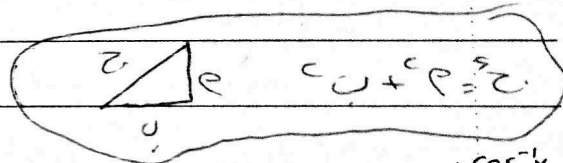
(شأنی) IR $\rightarrow$ IR

Example : find domain + range $f(x) = \frac{\pi}{2} + \tan^{-1}x$

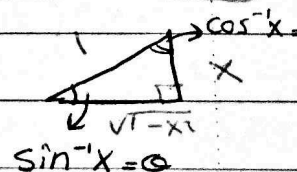
ايجاد هنا مجموع افترائين

$$Df = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

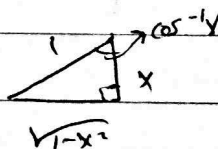
$$\underset{+\frac{\pi}{2}}{-\frac{\pi}{2}} < \underset{+\frac{\pi}{2}}{\tan^{-1}x} < \underset{+\frac{\pi}{2}}{\frac{\pi}{2}} \rightarrow 0 < \frac{\pi}{2} + \tan^{-1}x < \pi \quad Rf = (0, \pi)$$



$$\sin(\sin^{-1}x) = \frac{x}{1} = \frac{\text{المقابل}}{\text{الوتر}}$$

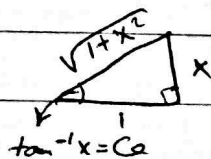


$$\cos(\cos^{-1}x) = \frac{x}{1} = \frac{\text{المجاور}}{\text{الوتر}}$$

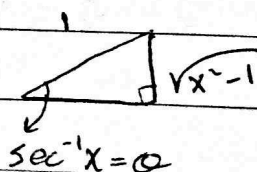


في حال فرضنا ان جوا
هو الزاوية

$$\tan(\tan^{-1}x) = \frac{x}{1} = \frac{\text{المقابل}}{\text{المجاور}}$$



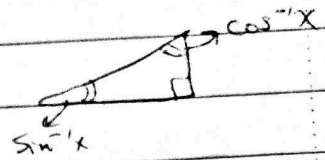
$$\sec(\sec^{-1}x) = \frac{x}{1} = \frac{\text{الوتر}}{\text{المجاور}}$$



Example:- Find

مجموع زاویه‌ها ۱۸۰

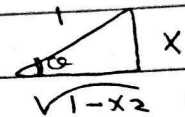
$$\textcircled{1} \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$



$$\textcircled{2} \tan(\sin^{-1}x) = \sin(\sin^{-1}x) = x = \frac{\text{مقابل}}{\text{وتر}}$$

← مقابل
← وتر

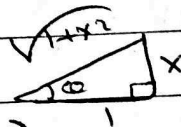
$$= \frac{x}{\sqrt{1-x^2}}$$



$$\textcircled{3} \sec(\tan^{-1}x) = \frac{1}{\cos(\tan^{-1}x)} = \frac{1}{\frac{1}{\sqrt{1+x^2}}} = \sqrt{1+x^2}$$

← وتر
← مجاور

$$= \sqrt{1+x^2}$$



$$\textcircled{4} \cos^2(\tan^{-1}x) = (\cos(\tan^{-1}x))^2$$

← مجاور
← وتر

$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

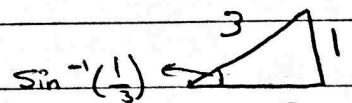
$$\cos^2 x = (\cos x)^2$$

$$\cos x^2 \neq \cos^2 x$$

$$\textcircled{5} \cot(\sin^{-1}(\frac{1}{3})) = \frac{\cos(\sin^{-1}(\frac{1}{3}))}{\sin(\sin^{-1}(\frac{1}{3}))} = \frac{1}{\frac{1}{3}} = 3 = \frac{\text{مقابل}}{\text{وتر}}$$

← مجاور
← مقابل

$$= \frac{\sqrt{8}}{1}$$

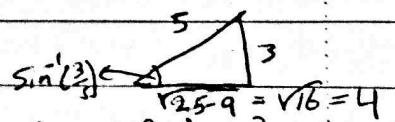


$$\sqrt{9-1} = \sqrt{8}$$

$$\textcircled{6} \sin(2\sin^{-1}(\frac{3}{5})) = 2\sin(\sin^{-1}(\frac{3}{5}))\cos(\sin^{-1}(\frac{3}{5}))$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$2(\frac{3}{5})(\frac{4}{5})$$



$$\sin(\sin^{-1}(\frac{3}{5})) = \frac{3}{5}$$

← مقابل
← وتر

H.W

$\sin(2a) = 2 \sin a \cos a$
 $\sin(a+b) = \sin a \cos b + \cos a \sin b$
 $\cos(a+b) = \cos a \cos b - \sin a \sin b$

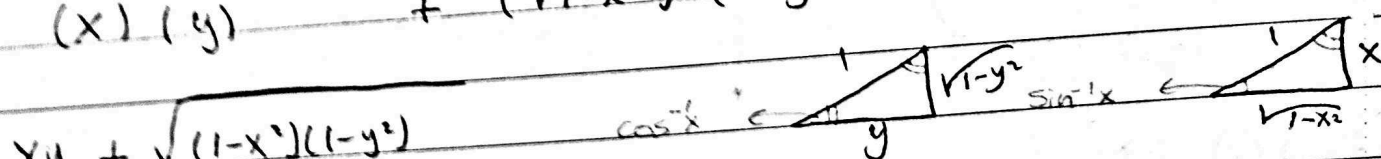
Note

at $\frac{\pi}{2}$ is 0 & π is π

(1) $\sin(\sin^{-1}x + \cos^{-1}y)$

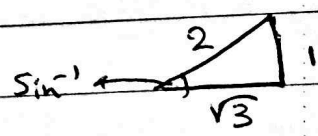
$(\sin(\sin^{-1}x))(\cos(\cos^{-1}y)) + (\cos(\sin^{-1}x))(\sin(\cos^{-1}y))$
 $(x)(y) + (\sqrt{1-x^2})(\sqrt{1-y^2}) =$

$xy + \sqrt{(1-x^2)(1-y^2)}$
 $= xy + \sqrt{1-y^2-x^2+xy^2} = \sqrt{x^2y^2-x^2-y^2+1} + xy$

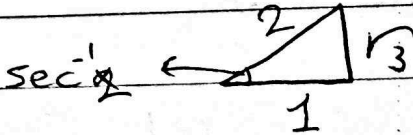


$= xy + \sqrt{1-y^2-x^2+xy^2} = \sqrt{x^2y^2-x^2-y^2+1} + xy$

(2) $\tan(\sin^{-1}(\frac{1}{\sqrt{3}})) = \frac{1}{\sqrt{3}}$



(3) $\sin(\sec^{-1}(2)) = \frac{\sqrt{3}}{2}$



Find the range (1) $f(x) = |7x-1| - 4$

$\text{Domain} \rightarrow \text{Range } f = [-4, \infty)$

(2) $f(x) = -|5x+8| + 9$

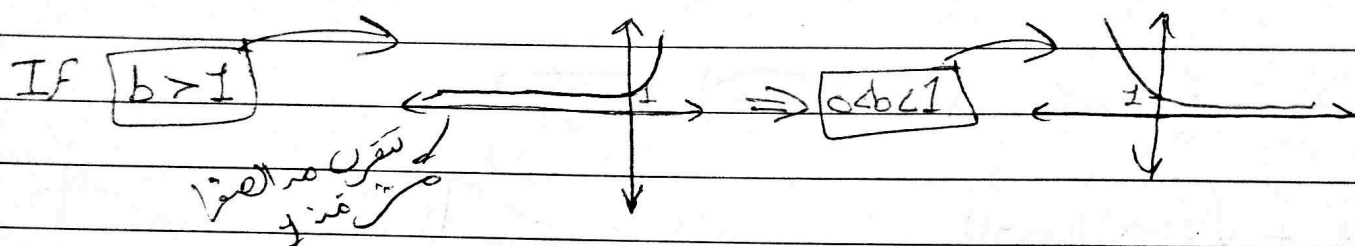
$(-\infty, 0]$
 $\text{Range } f = [-\infty, 9]$

Lecture 26

Exponential and logarithmic function:-

* Definition:- Exponential function

$$f(x) = b^x, \quad b > 0, \quad b \neq 1$$



Ex:- $f(x) = 3^x$

Ex:- $f(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$

Domain: $f = \mathbb{R}$

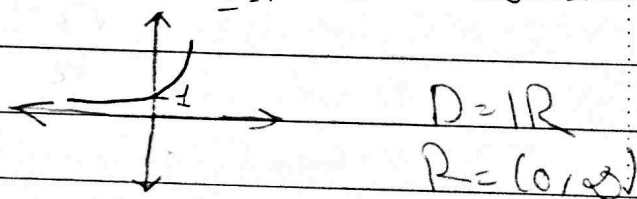
Range: $f = (0, \infty)$

Domain: $f = \mathbb{R}$

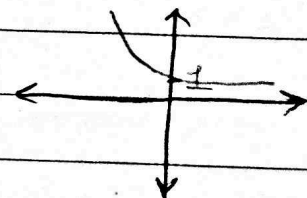
Range: $f = (0, \infty)$

* The natural exponential function:-

$f(x) = e^x$ $e \approx 2.718$



$f(x) = e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$ $\rightarrow$ أقل من واحد



Example:- Find domain

① $f(x) = e^{\sqrt{x}}$ $\rightarrow D_f = D_{\sqrt{x}} = [0, \infty)$
 $D(e^{g(x)}, b^{g(x)}) = D(g(x)) \rightarrow \text{Note}$

② $f(x) = \frac{x}{1+e^x} \rightarrow \mathbb{R}$ $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$
 $\mathbb{R} \xleftarrow{1+e^x} \mathbb{R}$

$e^x \rightarrow \neq 0$
 $\rightarrow \neq \infty$
 $1+e^x=0 \rightarrow e^x=-1$ (impossible, no solution)
 $\boxed{\neq} e^x = -1$

Lecture 27 Logarithmic function

$2^5 = 32 \Leftrightarrow \log_2 32 = 5$
 $8^1 = 8 \Leftrightarrow \log_8 8 = 1$

$\log_2 16 = 4 \rightarrow 2^4 = 16$ $x^y = z \rightarrow \log_x z = y$

Example:- Find the value $\log_a 3 = x$

$a^x = 3 \rightarrow (3^2)^x = 3 \rightarrow 3^{2x} = 3^1 = 2x=1 \rightarrow x = \frac{1}{2}$

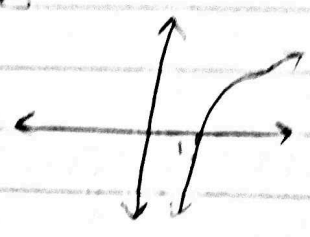
$\therefore \log_a 3 = \frac{1}{2}$

$f(x) = e^x$ $g(x) = \ln x$ $(f \circ g)(x) = f(g(x)) = e^{\ln x} = x$
 $g(x) = f^{-1}(x)$ $(g \circ f)(x) = g(f(x)) = \ln e^x = x$
 $F(x) = g^{-1}(x)$

$f(x) = \log_b x$, $b > 0$, $b \neq 1$, $x > 0$

$\log_b x = +$
 $0 \neq b$
 $x > 0$

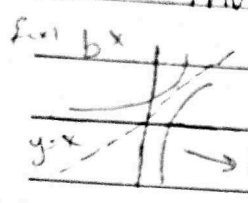
$\log_b 1 = 0$
 $\log_b b = 1$



$D_f = (0, \infty)$, $R = \mathbb{R}$

Theorem: If $b > 0$, $b \neq 1$ then $f(x) = b^x$, $f(x) = \log_b x$ are inverse

$x = \text{composition of } f \text{ and } g$



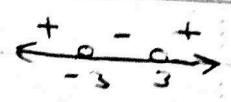
- * $\log_b x = \log x$ (common log)
- * $\log_e x = \ln x$ (natural log)

- Example:-** ① $\log_{10} 10 = 1$ ③ $\ln 1 = \log_e 1 = 0$
 ② $\ln e = \log_e e = 1$

Example:- Find domain

① $f(x) = \log(x^2 - 9)$

$(x-3)(x+3)$
 $x^2 - 9 > 0$



$D_f = (-\infty, -3) \cup (3, \infty)$

② $f(x) = \frac{1}{\ln(4-x)}$ $\ln(4-x) > 0$

$4-x > 0$

$4 > x$

$(-\infty, 4)$

$\ln(4-x) = 0 \rightarrow \log_e(4-x) = 0$

$e^0 = 4-x \rightarrow 1 = 4-x \rightarrow x = 4-1$

$x = 3$

$\leftarrow$ $f(3)$, $\ln(4-3) = \ln 1 = 0$

$\ln(4-x)$

$\ln(4-3) = \ln 1 = 0$

$D_f = \mathbb{R} \cap (-\infty, 4) \setminus \{3\}$

دون حل دستخط

$(-\infty, 4) \setminus \{3\} = (-\infty, 3) \cup (3, 4)$

Lecture 28

$$(f \circ f^{-1})(x) = x$$

$$(f^{-1} \circ f)(x) = x$$

* Logarithm Properties:

$$(1) b^{\log_b x} = x, e^{\ln x} = x$$

$$(2) \log_b b^x = x, \ln e^x = x$$

$$(3) \log_b a = \frac{\ln a}{\ln b}$$

$$(4) \log_b ac = \log_b a + \log_b c$$

$$\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

$$(5) \log_b \frac{a}{c} = \log_b a - \log_b c \rightarrow$$

$$\frac{\log a}{\log b} \neq \log_b a - \log_b c$$

$$(6) \log_b a^r = r \log_b a$$

$$\log x^5 = 5 \log x$$

$$(7) \log_b x = \log_b y \iff x = y$$

$$x = y$$

Example:- Find the values of x :-

(1)

$$(1) \log \sqrt{x} = 2 \rightarrow (10^2 = \sqrt{x})^2 = 10^4 = x$$

$$\log_{10} \sqrt{x} = 2$$

$$10 = 10 \rightarrow (\sqrt{x})^2 = (10^2)^2 = x = 10^4$$

$$(2) \log(x-2) + \log(2x-3) = 2 \log x$$

$$\log ac = \log a + \log c$$

$$\log[(x-2)(2x-3)] = \log x^2 \rightarrow (x-2)(2x-3) = x^2$$

$$2x^2 - 3x - 4x + 6 - x^2 = 0 \rightarrow x^2 - 7x + 6 = 0$$

$$(x-1)(x-6) = 0$$

$$x = 1, 6 \text{ [قبول]} \text{ [مقبول]}$$

$$\rightarrow \log x \rightarrow \begin{matrix} \neq 0 \\ \neq 1 \end{matrix}$$

$$b \rightarrow \begin{matrix} b > 0 \\ b \neq 1 \end{matrix}$$

$$\log(1-2) + \log(2-3) \neq \text{قبول}$$

$$1$$

$$x \in \{6\}$$

lecture 29

Example :- Find the Value of x

① $\log_2 4 = 2$ $X^2 = 4 \rightarrow X^2 = 2^2$ $(X=2)$ $(X=2)$
 $\neq 1$ or $\sqrt{X^2} = \sqrt{4} \rightarrow |X| = 2 \rightarrow X = \pm 2 \rightarrow X \neq 2$
 مخرج

② $5^x = 7$ $\log_5 5^x = \log_5 7$ $\log_b b^x = X$
 $X = \log_5 7$ $\ln 5^x = \ln 7 \rightarrow x \frac{\ln 5}{\ln 5} = \frac{\ln 7}{\ln 5}$

$\log_b a = \frac{\ln a}{\ln b}$

3

$X = \log_5 7$

مخرج الجواب

log

③ $(x^2-1)(x-5) \cdot x^3 \cdot \log_2 x = 0$ $abc=0$
 $(x^2-1)=0 \rightarrow X = \pm 1$ $(x-5)=0 \rightarrow X=5$ $\checkmark$
 $x^3=0 \rightarrow X=0$ $\log_2 x=0 \rightarrow x^0=2x \rightarrow 2x=1 \rightarrow X=\frac{1}{2}$ $\checkmark$
 $X = [5, \frac{1}{2}]$

④ $e^{2x} - 6e^x - 7 = 0 \rightarrow (e^x)^2 - 6e^x - 7 = 0$ $(let y=e^x)$
 $y^2 - 6y - 7 = 0$ $(y+1)(y-7) = 0$ $y = -1$ $(e^x \neq -1)$
 $y = 7 \rightarrow e^x = 7 \rightarrow \ln e^x = \ln 7 \rightarrow X = \ln 7$

$\log_2 3x + \log_4 9x^2 = 4 \rightarrow \frac{\ln 3x}{\ln 2} + \frac{\ln 9x^2}{\ln 4} = 4 \rightarrow \frac{2 \ln 3x}{2 \ln 2} + \frac{\ln 9x^2}{2 \ln 2} = 4$

$\ln 9x^2 + \ln 9x^2 = 8 \ln 2 \rightarrow 2 \ln 9x^2 = 8 \ln 2 \rightarrow \ln 9x^2 = 4 \ln 2$
 $\ln 9x^2 = \ln 2^4 \rightarrow 9x^2 = 2^4 \rightarrow X = \sqrt{\frac{16}{9}} \rightarrow X = \pm \frac{4}{3}$

$$\log_a a + \log_b b = \log_c c$$

$$⑤ \quad 1 - 2 \ln x < 3$$

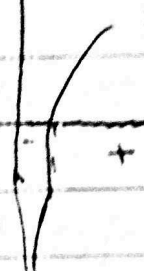
$$\frac{-2 \ln x < 2}{-2} \rightarrow \ln x > -1 \rightarrow x > e^{-1} \rightarrow x > \frac{1}{e}$$

$$x \in (e^{-1}, \infty) = (\frac{1}{e}, \infty)$$

استدلال (log)

$$\ln x > 0 \rightarrow x > 1$$

$$\ln x < 0 \rightarrow 0 < x < 1$$



$$\frac{\ln 2x}{\ln 2} > \frac{5}{\ln 2} \rightarrow x > \frac{5}{\ln 2}$$

2 ln صيغة
مقابل (7, 4)

لو كانت e هنا لما ينطبق لأنه اقرب من $\left[\frac{1}{2}\right]$ فمجرد e فيمكن الإشارة الكافية

Example:- $f(x) = \ln(x-1) + 5$ find $f^{-1}(x)$?

$$y = \ln(x-1) + 5 \rightarrow y - 5 = \ln(x-1)$$

$$e^{y-5} = x-1 \rightarrow e^{y-5} + 1 = x \rightarrow e^{x-5} + 1 = f^{-1}(x)$$

Example H.w Find the exact value of

$$① \log_2 8 + \log_4 2 = \frac{\ln 8}{\ln 2} + \frac{\ln 2}{\ln 4} \rightarrow \frac{\ln 8}{\ln 2} + \frac{\ln 2}{\ln 4} = \frac{\ln 8}{\ln 2} + \frac{\ln 2}{\ln 2^2} = \frac{\ln 8}{\ln 2} + \frac{1}{2}$$

$$② \ln(e^b) \rightarrow \ln(e^b) = b \quad \frac{\ln 2^3}{\ln 2} + \frac{\ln x}{\ln 2^2} = 3 + \frac{1}{2}$$

$$(e^x)^2 \neq e^{x^2}$$

$$\text{let } y = e^x$$

$$(1) \frac{e^{2x} - 1}{e^x} = 2 \rightarrow e^{2x} - 1 = 2e^x \rightarrow e^{2x} - 2e^x - 1 = 0$$

$$y^2 - 2y - 1 = 0 \quad y = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} \quad \sqrt{8} = 2\sqrt{2}$$

$$y = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \quad \ln e^x = \ln(1 + \sqrt{2}) \rightarrow x = \ln(1 + \sqrt{2})$$

$$\ln e^x = \ln(1 - \sqrt{2}) \rightarrow x \neq \ln(1 - \sqrt{2})$$

$$\text{OR } \frac{e^x - 1}{e^x} = 2 \rightarrow e^{2x} - 2e^x - 1$$

$$\sqrt{2} > \sqrt{1} \quad \text{w/ a.s.} \quad (1 - \sqrt{2} < 0) \quad (u.p.)$$

$$(2) 9(3)^x = 4^{-x}$$

$$3^2(3)^x = 4^{-x} \rightarrow \ln 3^{2+x} = \ln 4^{-x} \rightarrow (2+x) \ln 3 = -x \ln 4$$

$$2 \ln 3 + x \ln 3 = -x \ln 4 \rightarrow 2 \ln 3 = -x (\ln 4 - \ln 3)$$

$$\frac{2 \ln 3}{(-\ln 4 - \ln 3)} = x$$

$$\text{Find range } f(x) = 4 + 2 \cos x + |\cos x| \quad -1 \leq \cos x \leq 1$$

$$\text{max} \rightarrow 4 + 2(1) + |1| = 4 + 2 + 1 = 7$$

$$\text{min} \rightarrow 4 + 2(-1) + |-1| = 4 - 2 + 1 = 3 \quad \text{range } f \quad [3, 7]$$

$$3 \leq f(x) \leq 7$$

$$\text{Find } f'(x) \text{ for } f(x) = \frac{(x+1)^3}{x^3}, x \neq 0$$

$$y = \frac{(x+1)^3}{x^3} \rightarrow \sqrt[3]{y} = \sqrt[3]{\left(\frac{x+1}{x}\right)^3} \rightarrow \sqrt[3]{y} = \frac{x+1}{x} \rightarrow x \sqrt[3]{y} - x = 1$$

$$x(\sqrt[3]{y} - 1) = 1 \rightarrow x = \frac{1}{\sqrt[3]{y} - 1} \rightarrow f'(x) = \frac{1}{3\sqrt[3]{x} - 1}$$

$$f(x) = \frac{\pi}{2} + \tan^{-1} x$$

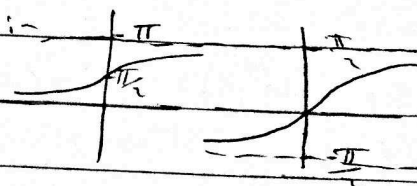
Find Domain R in

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

DRAR - IR

$$0 < \tan^{-1} x + \frac{\pi}{2} < \pi$$

R = (0, \pi)



Lecture 30

The limit of a function

(نفايات)

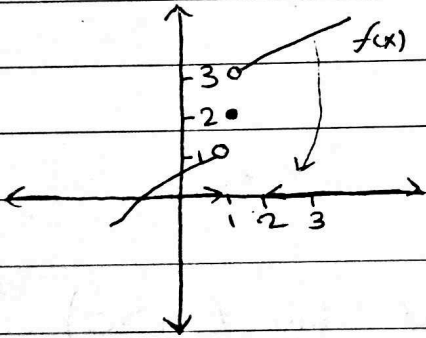
* Definition:- Suppose $f(x)$ is defined when x is near the number a then we write

$\lim_{x \rightarrow a} f(x) = L$ and say "the limit of $f(x)$ as x goes to a , equals L "

* If $\lim_{x \rightarrow a^+} f(x) = L$ and $\lim_{x \rightarrow a^-} f(x) = L$ then $\lim_{x \rightarrow a} f(x) = L$ exist
 نهاية غير موجودة

* If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ then $\lim_{x \rightarrow a} f(x)$ does not exist (d.n.e)

★ Example:-



find

① $f(1) = 2$

② $\lim_{x \rightarrow 1^+} f(x) = 3$

$$\textcircled{3} \lim_{x \rightarrow i^-} f(x) = 1$$

$$\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x)$$

④ $\lim_{x \rightarrow 1} f(x) = \text{d.n.e}$

* Limit laws:- Suppose that $[c]$ is a constant and the limits $\lim_{x \rightarrow a} f(x)$, $\lim_{x \rightarrow a} g(x)$ exist Then

$$\textcircled{1} \lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) \quad \text{توزيع على الجمع والطرح}$$

$$\textcircled{2} \lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$$

$$\textcircled{3} \lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) \quad \text{توزيع على الضرب}$$

$$\textcircled{4} \lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0 \quad \text{توزيع على القسمة}$$

$$\textcircled{5} \lim_{x \rightarrow a} c = c \quad \text{ثابتة} \quad \text{ثابتة} = \text{ثابتة}$$

$$\textcircled{6} \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad n \text{ Positive integer}$$

(If n even we assume that $\lim_{x \rightarrow a} f(x) > 0$) موجب

$$\textcircled{7} \lim_{x \rightarrow a} f(x) = f(a) \quad \text{If } f \text{ is a polynomial or a rational function and } a \in D_f \quad (*) \text{ (a polynomial, rational)} \quad (*)$$

$$\textcircled{8} \text{ If } f(x) \leq g(x) \text{ then } \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \rightarrow \begin{array}{|l} g(x) \\ \hline f(x) \end{array}$$

Example :- Find

$$\textcircled{1} \lim_{x \rightarrow 5} 9 = 9$$

$$\textcircled{2} \lim_{x \rightarrow 1} (x^3 - 2x + 1) = f(1) \rightarrow 1^3 - 2 + 1 = 0$$

lecture 3

$$\lim_{x \rightarrow a} \frac{f}{g} \rightarrow \frac{\lim_{x \rightarrow a} f}{\lim_{x \rightarrow a} g}$$

$$\frac{a+c}{b+d} = \frac{ad+cb}{bd}$$

$$\lim_{x \rightarrow a} \frac{f}{g} = \frac{c}{d}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

Example: $\lim_{x \rightarrow -2} \frac{2x+4}{3x-2} = \frac{0}{-8} = 0$

$$\lim_{x \rightarrow a} \frac{f}{g} = \frac{\text{num}}{\text{den}}$$

إعادة تعريف

فيل

التعريف

$$\lim_{x \rightarrow a} \frac{f}{g} = \frac{\text{num}}{\text{den}}$$

صديق مرافق

السمة المتكافئة أو البديلة

Example: Find (1) $\lim_{x \rightarrow -4} \frac{2x+8}{x^2+x-12} = \frac{0}{16-4-12} = \frac{0}{0}$

$$\frac{2(x+4)}{(x+4)(x-3)} = \frac{2}{x-3} = \frac{2}{-4-3} = \frac{-2}{7}$$

$$\lim_{x \rightarrow 0} \frac{x}{1-\sqrt{x+1}} \left(\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} \right)$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(\sqrt{x}+1)(\sqrt{x}-1) = x-1$$

$$(2-\sqrt{x})(2+\sqrt{x}) = 4-x$$

$$\lim_{x \rightarrow 0} \frac{x(1+\sqrt{x+1})}{1-(x+1)} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{x+1})}{1-x-1} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{x+1})}{-x} = -1 - \sqrt{x+1}$$

$$-1 - \sqrt{0+1} = -1 - 1 = -2$$

(3)

$$\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \frac{0}{0} = \frac{x+4}{4x} = \frac{(x+4)}{4x(4+x)} = \frac{1}{4x} = \frac{1}{-16}$$

الحل النهائي

Lecture 32

Example: Find

$$\textcircled{1} \lim_{x \rightarrow 1} \frac{\sqrt[4]{x+15} - 2}{x-1} = \frac{0}{0} \quad \left(\text{let } y = \sqrt[4]{x+15} \Rightarrow y^4 = x+15 \rightarrow y^4 - 15 = x \right)$$

$x \rightarrow 1, y = 2$

$$\lim_{y \rightarrow 2} \frac{y-2}{y^4-15-1}$$

لما بنيل 8 في 8 كل واحد واحد بنيل 64 في 64

$$\lim_{y \rightarrow 2} \frac{y-2}{y^4-16} = \lim_{y \rightarrow 2} \frac{y-2}{(y^2+4)(y^2-4)} = \frac{y-2}{(y^2+4)(y-2)(y+2)}$$

$$\frac{1}{(y^2+4)(y+2)} = \frac{1}{(8)(4)} = \frac{1}{32}$$

$$\textcircled{2} \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$x=0 \quad \begin{array}{c} -x \quad x \\ \frac{-x}{x} \quad 0 \quad \frac{x}{x} \end{array}$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

$$\therefore \lim_{x \rightarrow 0} \frac{|x|}{x} = \text{d.n.e}$$

$$\textcircled{3} \lim_{x \rightarrow 2} \frac{|x-3|}{2x-6} = \frac{|2-3|}{4-6} = \frac{1}{-2}$$

$$\textcircled{4} \lim_{x \rightarrow 3^+} \frac{|x-3|}{2x-6} = \lim_{x \rightarrow 3^+} \frac{x-3}{2x-6}$$

$$x-3=0 \quad x=3 \quad \begin{array}{c} 3-x \quad x-3 \\ \frac{3-x}{3} \quad \frac{x-3}{3} \end{array}$$

How $\lim_{x \rightarrow 3^+} \frac{x-3}{2(x-3)} = \frac{1}{2}$

$$\lim_{x \rightarrow 3^-} \frac{3-x}{2x-6} = \lim_{x \rightarrow 3^-} \frac{(3-x)-1}{2(x-3)} = \frac{-1}{2}$$

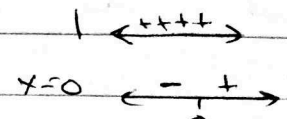
$$\lim_{x \rightarrow 3} \frac{|x-3|}{2x-6} = \text{d.n.e}$$

Lecture 33

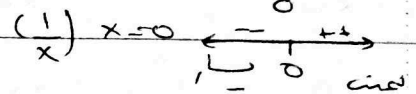
$$\lim_{x \rightarrow a} \frac{f}{g} = \frac{\text{num}}{\text{den}} \begin{matrix} +\infty \\ -\infty \\ \text{d.n.e} \end{matrix}$$

Example :- Find

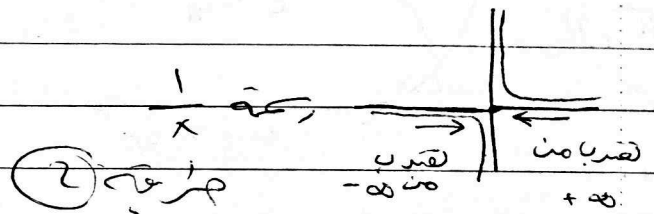
$$\textcircled{1} \lim_{x \rightarrow 0^+} \frac{1}{x} \rightarrow \frac{1}{0} \rightarrow \text{نصف الاصل} = +\infty$$



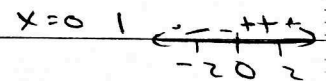
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty \quad \textcircled{1} \text{ طرف}$$



$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ d.n.e}$$

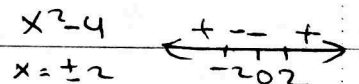


$$\textcircled{2} \lim_{x \rightarrow 2} \frac{x}{x^2-4} = \frac{2}{0} \rightarrow \text{نصف الاصل}$$



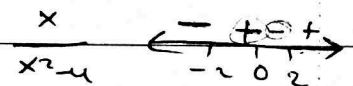
$$\lim_{x \rightarrow 2^+} \frac{x}{x^2-4} = +\infty$$

$$\lim_{x \rightarrow 2^-} \frac{x}{x^2-4} = \frac{0}{-4} = 0$$



$$\lim_{x \rightarrow 2^-} \frac{x}{x^2-4} = -\infty$$

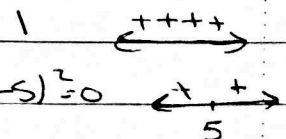
d.n.e في 2



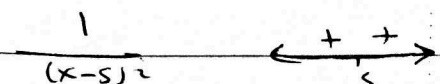
نصف الاصل

$$\lim_{x \rightarrow 2} \frac{x}{x^2-4} = \text{d.n.e}$$

$$\textcircled{3} \lim_{x \rightarrow 5^-} \frac{1}{(x-5)^2} \xrightarrow{+\infty} \lim_{x \rightarrow 5^+} \frac{1}{(x-5)^2} = +\infty \quad x=5 \leftarrow (x-5)^2=0$$



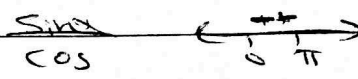
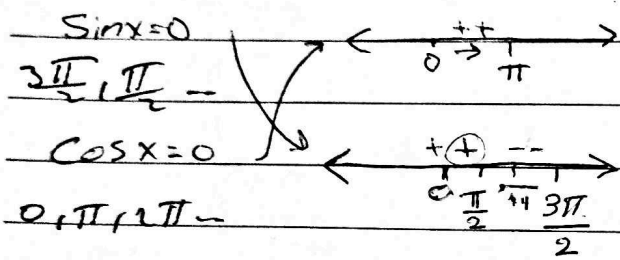
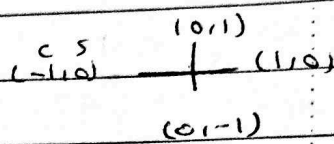
$$\lim_{x \rightarrow 5} \frac{1}{(x-5)^2} = +\infty$$



10



$$(5) \lim_{x \rightarrow \pi^-} \cot x = \lim_{x \rightarrow \pi^-} \frac{\cos x}{\sin x} = +\infty$$



lecture 34

تقارب عمودي

Definitions: The line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following statements is true

يكون صحيح في واحد من الحالات

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

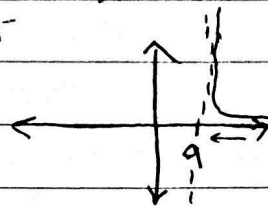
$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} = +\infty$$



Example: Find the vertical asymptote of

$$f(x) = \frac{x-1}{x^2-1}$$

$$x = \pm 1$$

$$x^2 - 1 = 0 \Rightarrow x = 1, -1$$

at $x=1$:-

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{(x-1)}{(x-1)(x+1)} = \frac{1}{2}$$

at $x=-1$

$$\lim_{x \rightarrow -1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow -1} \frac{-2}{0} = \text{D.N.E.}$$

Vertical asymptote at $x=-1$

$$\lim_{x \rightarrow -1^+} \frac{x-1}{x^2-1} = +\infty$$

$$\lim_{x \rightarrow -1^-} \frac{x-1}{x^2-1} = -\infty$$

$$f(x) = \frac{x-2}{|x|-2}$$

$$|x|-2 = 0 \Rightarrow x = \pm 2$$

find limit

at $x=2$

$$\lim_{x \rightarrow 2} \frac{x-2}{|x|-2} = \frac{x-2}{x-2} = 1$$

at $x=-2$

$$\lim_{x \rightarrow -2} \frac{x-2}{|x|-2} = \frac{-4}{0}$$

$$\lim_{x \rightarrow -2} \frac{x-2}{-x-2} = \frac{-4}{-4} = 1$$

$$\lim_{x \rightarrow -2^+} = +\infty$$

$$\lim_{x \rightarrow -2^-} = +\infty$$

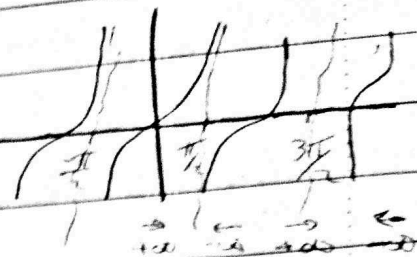
$$(3) f(x) = \tan x = \frac{\sin x}{\cos x} \quad \text{S.P.}$$

now $\sin x = 0, \pi, 2\pi, 3\pi$

$$\cos x = 0 \quad x = \pm \pi, \pm 3\pi, \pm \frac{5\pi}{2}$$

$$\text{at } x = \pm \pi, \pm 3\pi$$

$$x = \pm (2n+1) \frac{\pi}{2}$$



$$(4) f(x) = \ln x$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$\therefore x = 0$ vertical asymptote

$$(5) f(x) = \ln(x^2 - 9)$$

$$x^2 - 9 = 0 \rightarrow$$

$$x = \pm 3$$

vertical asymptote

at $x = 3, x = -3$ vertical asymptote

How

$$(6) f(x) = \cot x = \frac{\cos x}{\sin x}$$

$$\sin x = 0 \left[\pm \pi, \pm 2\pi, \pm 3\pi \right]$$

Lecture 35

نسب الامكان الحساب

نقطة

$$f(x) = \begin{cases} \sqrt{x-4} & x > 4 \\ 8-2x & x \leq 4 \end{cases}$$

Find
 $\xleftarrow{8-2x} \quad \xrightarrow{\sqrt{x-4}}$
 (نموذج مباشر) عادية (نقطة) $x=4$ $\rightarrow$ قول او تشعب

$$\textcircled{1} \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (8-2x) = \text{من المصنف او من اليسار}$$

$$8-2(0)=8$$

$$\textcircled{2} \lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4+} \sqrt{x-4} = 0 \quad \lim_{x \rightarrow 4-} 8-2x = 0$$

النهاية موجودة

Example: $f(x) = \begin{cases} 5, & x \leq 2 \\ x+a, & x > 2 \end{cases}$ Find a if $\lim_{x \rightarrow 2} f(x)$ exist?

$$\lim_{x \rightarrow 2+} (x+a) = \lim_{x \rightarrow 2+} 5 \rightarrow 5 = 2+a \quad \boxed{a=3}$$

Example: $g(x) = \begin{cases} x+1, & x \neq 1 \\ \pi, & x = 1 \end{cases}$ Find $\lim_{x \rightarrow 1} g(x)$?

$$\lim_{x \rightarrow 1+} x+1 = \lim_{x \rightarrow 1-} x+1 = \lim_{x \rightarrow 1} g(x) = 1+1 = 2$$

Example: Find

عنة الكونور في المجال قبل ايجاد النهاية

$$\textcircled{1} \lim_{x \rightarrow 0^-} \sqrt{x} \text{ d.n.e} \quad \textcircled{2} \lim_{x \rightarrow 1} \sqrt{x} = \sqrt{1} \quad \text{بالمستقيم} \quad [0, \infty)$$

$$\textcircled{3} \lim_{x \rightarrow 7^+} \sqrt{x-7} = \sqrt{7-7} = 0 \quad \textcircled{4} \lim_{x \rightarrow 2} \sqrt{x-7} \text{ d.n.e} \quad [7, \infty)$$

Theorem: $\lim_{x \rightarrow 0} \frac{\sin(ax)}{bx} = \lim_{x \rightarrow 0} \frac{ax}{\sin(bx)} = \frac{a}{b}$

$$\lim_{x \rightarrow 0} \frac{\tan(ax)}{bx} = \lim_{x \rightarrow 0} \frac{ax}{\sin(bx)} = \frac{a}{b}$$

Example: Find

$$1) \lim_{x \rightarrow 0} \frac{x}{\cos x} = \frac{0}{1} = 0$$

$$2) \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0} = \frac{1}{1} = 1$$

$$3) \lim_{x \rightarrow 0} \frac{\tan 4x}{5x} = \frac{4}{5}$$

$$4) \lim_{x \rightarrow 0} \frac{x \sin x - \tan(2x)}{\sin(3x) - 5x} \stackrel{0/0}{=} \lim_{x \rightarrow 0} \frac{x' \sin x - \tan(2x)}{\sin(3x) - 5x}$$

$$\frac{\sin(3x)}{x} - \frac{5x}{x}$$

$$\frac{0 - 2}{3 - 5} = \frac{-2}{-2} = +1$$

$$5) \lim_{x \rightarrow 5} \frac{\sin(x-5)}{x-5}$$

Let $y = x - 5$, $x \rightarrow 5$
 $y \rightarrow 0$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$6) \lim_{x \rightarrow 5} \frac{\sin(x^2 - 25)}{(x-5)} \left(\frac{x+5}{x+5} \right) = \lim_{x \rightarrow 5} (x+5) \cdot \lim_{x \rightarrow 5} \frac{\sin(x^2 - 25)}{(x-5)(x+5)}$$

$$\lim_{x \rightarrow 5} (x+5) \cdot \lim_{x \rightarrow 5} \frac{\sin(x^2 - 25)}{(x^2 - 25)} = 10 \cdot \lim_{y \rightarrow 0} \frac{\sin y}{y} = 10 \cdot 1 = 10$$

$$y = x^2 - 25$$

$$x \rightarrow 5$$

 $y \rightarrow 0$

lecture

The Squeeze theorem :- $\lim_{x \rightarrow a} f(x) = L$ if $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \rightarrow a} g(x) = L$ and $\lim_{x \rightarrow a} h(x) = L$

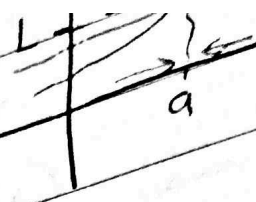
then $\lim_{x \rightarrow a} f(x) = L$

Example: - If $\lim_{x \rightarrow 1} f(x) = ?$

$\lim_{x \rightarrow 1} 3x = 3$, $\lim_{x \rightarrow 1} x^3 + 2 = 3$

$\therefore \lim_{x \rightarrow 1} f(x) = 3$

by Squeeze theorem



(ايب) Show that $\lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x}) = 0$?

Example: - Find

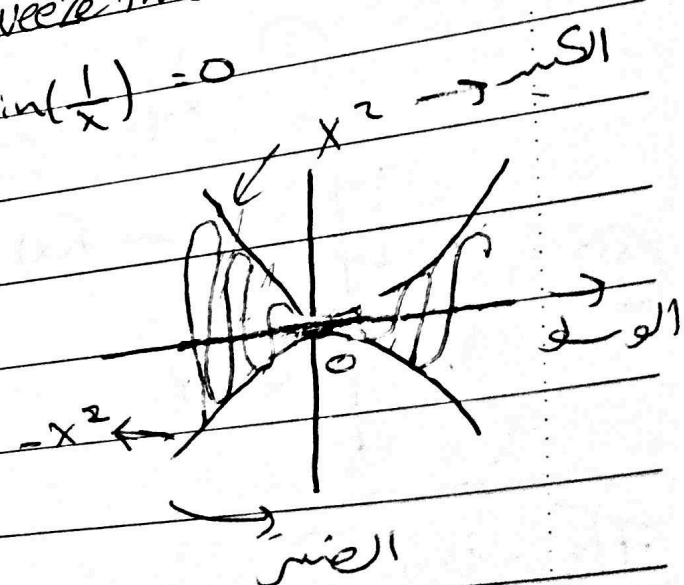
$-1 \leq \sin(\frac{1}{x}) \leq 1$

$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$

by Squeeze theorem

$\lim_{x \rightarrow 0} -x^2 = 0$, $\lim_{x \rightarrow 0} x^2 = 0$

$\therefore \lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x}) = 0$



lecture 37

* continuity

Definition: continuity at number

f is continuous at $(x=a)$ if

① $f(a)$ defined

② $\lim_{x \rightarrow a} f(x)$ exist

③ $\lim_{x \rightarrow a} f(x) = f(a)$

Example: Is $f(x)$ continuous at $x=3$

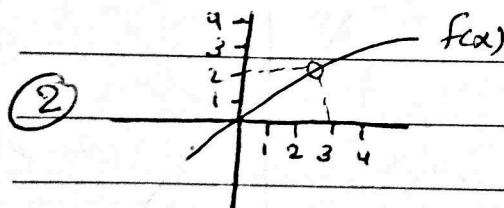
① $f(x) = \frac{x^2-9}{x-3}, x \neq 3$

6 $\circ$ $x=3$

6
~ $\uparrow$ ~
3

① $f(3) = 6$ ② $\lim_{x \rightarrow 3} \frac{x^2-9}{x-3} = 6$ $f(3) = \lim_{x \rightarrow 3} f(x) \therefore f$ con at $x=3$

$(x/3)(x+3)$



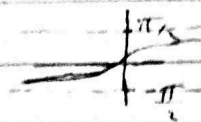
② $f(3)$ is undefined $\therefore f$ not con (discon) at $x=3$

Theorem:-

The following types of function are continuous at every number in these domains: Polynomials, rational function, trigonometric function, inverse trigonometric functions, exponential function, logarithmic function.

Example where f is continuous

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 9} \rightarrow \begin{matrix} (0, \infty) \cap \mathbb{R} & \mathbb{R} \\ & (0, \infty) \\ & \mathbb{R} \end{matrix}$$



$$Df = (0, \infty) \cap \mathbb{R} \setminus \{\pm 3\} = (0, \infty) \setminus \{3\}$$

پیدا/میل

$$x^2 - 9 = 0 \rightarrow x = \pm 3$$

f cont on $(0, \infty) \setminus \{3\}$

← $\frac{0}{0}$ ←

اذا ان غير مستمر ونب $(-\infty, 0] \cup \{3\}$

Example: Find k such that f continuous everywhere

(on $\mathbb{R}$)

كذلك

$$\textcircled{1} f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x+k, & x > 2 \end{cases}$$

$$\therefore f \text{ con at } x=2 \Rightarrow \lim_{x \rightarrow 2} f \text{ exist} \Rightarrow \lim_{x \rightarrow 2^+} 2x+k = \lim_{x \rightarrow 2^-} kx^2$$

$$2(2)+k = k(2)^2 \rightarrow 4+k = 4k \rightarrow 4=3k \rightarrow k = \frac{4}{3}$$

$$\textcircled{2} f(x) = \begin{cases} \frac{\sin(kx)}{5x}, & x < 0 \\ 2, & x > 0 \end{cases} \quad F \text{ con at } x=0$$

$$\lim_{x \rightarrow 0^+} 2 = \lim_{x \rightarrow 0^-} \frac{\sin(kx)}{5x} \rightarrow 2 = \frac{k}{5} \rightarrow k=10$$

Example:- where f is not continuous (discontinuous) غير مستمر

$$\textcircled{1} f(x) = \frac{1}{\sqrt{x-2}} \rightarrow \begin{matrix} \mathbb{R} \\ (2, \infty) \end{matrix} \quad Df = (2, \infty) \leftarrow \frac{\text{cont}}{2} \quad x=2 \text{ غير قابل}$$

$$\text{Answer:- } (-\infty, 2] = \mathbb{R} \setminus (2, \infty)$$

$$② f(x) = \begin{cases} 2x+3, & x < 4 \\ 7 + \frac{16}{x+4}, & x > 4 \end{cases}$$

$$\xleftarrow{2x+3} \quad \xrightarrow{7+\frac{16}{x+4}} \\ 4$$

$(-\infty, 4) \rightarrow 2x+3$ poly cont

$(4, \infty) \rightarrow 7 + \frac{16}{x+4}$ cont on $\mathbb{R} \setminus \{4\}$ cont $(4, \infty)$

* at $x=4$

$$f(4) = 2(4)+3 = 8+3 = 11$$

$$\lim_{x \rightarrow 4^+} \left(7 + \frac{16}{x+4} \right) = 7 + \frac{16}{8} = 9$$

$$\lim_{x \rightarrow 4^+} f \neq f(4) \quad \therefore f \text{ not cont at } x=4$$

f cont on $\mathbb{R} \setminus \{4\}$

$\therefore f$ not cont (discont) at $x=4$

Lecture 38

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \lim_{x \rightarrow b^-} f(x) = f(b)$$

$$\begin{matrix} k & & l \\ a & \rightarrow & c & \leftarrow & b \end{matrix}$$

نقطة النهاية عند

① الفترات المفتوحة

② عند الأطراف

③ نقاط التوقف

continuity at an interval :- الاتصال بفترة

f is continuous on $[a, b]$ if

- ① f continuous on (a, b)
- ② $\lim_{x \rightarrow a^+} f(x) = f(a)$
- ③ $\lim_{x \rightarrow b^-} f(x) = f(b)$

* Example :- $f(x) = \begin{cases} x^2 - 5x, & 0 \leq x < 1 \\ \frac{1}{x}, & 1 \leq x \leq 4 \end{cases}$

is f continuous on $[0, 4]$?

$(0, 1) \rightarrow f(x) = x^2 - 5x$ poly cont on $\mathbb{R}$, cont $(0, 1)$

$(1, 4) \rightarrow f(x) = \frac{1}{x}$, cont on $\mathbb{R} \setminus \{0\}$, $0 \notin (1, 4)$

$x^2 - 5x$	$\frac{1}{x}$
0	1 4

f cont on $(1, 4)$

at $x=1$ ① $f(x) = 1$ ② $\lim_{x \rightarrow 1^+} \frac{1}{x} = 1$ ③ $\lim_{x \rightarrow 1^-} = -4$

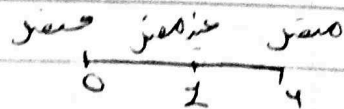
f not cont at 1

u/p 81

at $x=0$

① $f(0)=0$ ② $\lim_{x \rightarrow 0} (x^2 - 5x) = 0$ $\therefore f$ cont at $x=0$

③ $\lim_{x \rightarrow 0^+} f = f(0)$



at $x=4$

① $f(4) = \frac{1}{4}$ ② $\lim_{x \rightarrow 4^-} \frac{1}{x} = \frac{1}{4}$ ③ $\lim_{x \rightarrow 4^-} f = f(4)$

$\therefore f$ cont at $x=4$

$\therefore f$ cont on $[0, 4] \setminus [1, 1]$ dis(cont) = $x=1$

Theorem:- If f and g are Continuous at a and C is a constant, then the following functions are also continuous at $x=a$:-

- ① $f+g$ ② $f-g$ ③ cf ④ $f \cdot g$ ⑤ $\frac{f}{g}$ if $g(a) \neq 0$
 $\hookrightarrow$ ① $(f+g)(a)$ exist ② ~ ③ ~

Example :-

$f(x) = 5x^2$

* Qs $\frac{f}{g}$ continuous at $x=2$?

$g(x) = x-1$

* $f(x) = 5x^2$ Poly cont on $(\mathbb{R})$, cont at $x=2$

$g(x) = x-1$ Poly ~ $(\mathbb{R})$

by theorem $\frac{f}{g}$ cont at $x=2$, $g(2) \neq 0 \neq 1$ *

* $\frac{f}{g} = \frac{5x^2}{x-1}$ ① $\frac{f}{g}(2)$ ② $\lim_{x \rightarrow 2} \frac{5x^2}{x-1}$ ③ =

* $\frac{f}{g} = \frac{5x^2}{x-1}$ cont on $\mathbb{R} \setminus [1, 1]$ $\therefore \frac{f}{g}$ cont at $x=2$

Lecture 39

اشتقاق

Example:-

① Find $\frac{dy}{dx}$ for

② بالعربية X

② $f(x) = \sqrt{x^x}$ $y = (x^x)^{1/2} = x^{x/2}$ $\ln y = \ln(x^{x/2})$

$\ln y = \frac{x}{2} \ln x \rightarrow$

$(\log)^{(x)} = f(g(x))$

* Theorem: If ① $\lim_{x \rightarrow c} g(x) = L$ ② f continuous at $x = L$ then

$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x)) = f(L)$

Example:-

① $\lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) \rightarrow \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x}\right)$

$f = \sin^{-1} x$

$= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}\right) = \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-x}{1-x} \cdot \left(\frac{1}{1+\sqrt{x}}\right)\right)$

$g = \frac{1-\sqrt{x}}{1-x}$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\sqrt{[-1, 1]}$ $\frac{1}{2}$ $\frac{1}{2}$

② $\lim_{x \rightarrow 1} e^{\frac{x^2-1}{x-1}} = e^{\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}} = e^{\lim_{x \rightarrow 1} (x+1)} = e^2$

$f = e^x$

$g = \frac{x^2-1}{x-1}$

$$\textcircled{3} \lim_{x \rightarrow 1} \cos\left(\frac{x^2-1}{x-1}\right) = \cos\left(\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}\right)$$

$$\cos(2) = 0.999$$

Theorem: If $\textcircled{1} g$ continuous at $x=c$ $\textcircled{2} f$ continuous at $x=g(c)$
 then $f \circ g$ continuous at $x=c$ $f \circ g$ cont at c

Example: $f(x) = \frac{1}{x - \frac{1}{2}} \rightarrow$ cont on $\mathbb{R} \setminus [\frac{1}{2}]$

$$g(x) = \frac{1}{2x} \rightarrow \text{cont on } \mathbb{R} \setminus \{0\}$$

$\textcircled{1}$ Is $f \circ g$ continuous at $x=0$? $\textcircled{1} g$ cont at $x=0$? $\textcircled{2} \therefore g$ not cont at $x=0$

$\therefore f \circ g$ not cont at $x=0$

$\textcircled{2}$ Is $f \circ g$ continuous at $x=\frac{1}{2}$? $\textcircled{1} g$ cont at $x=\frac{1}{2}$? yes g cont at $x=\frac{1}{2}$

$\textcircled{2} f$ cont at $x=g(\frac{1}{2}) = \frac{1}{2}$? NO $\therefore f$ not cont at $x=\frac{1}{2}$

$\therefore f \circ g$ not cont at $x=\frac{1}{2}$

$\textcircled{3}$ Is $f \circ g$ continuous at $x=2$? $\textcircled{1} g$ cont at $x=2$? yes g cont at $x=2$

$\textcircled{2} f$ cont at $x=g(2) = \frac{1}{4}$? yes f cont at $x=\frac{1}{4}$

$\therefore f \circ g$ cont at $x=2$

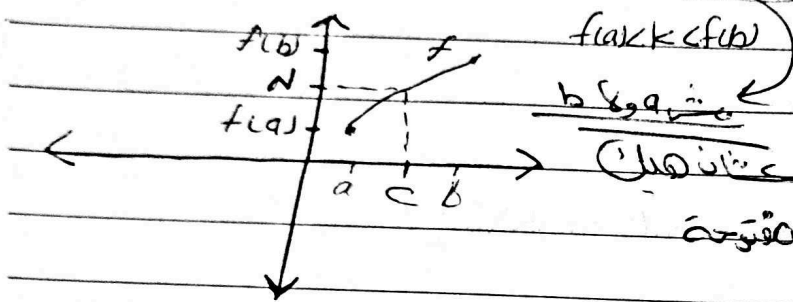
Lecture 40

المركبة

is The intermediate value theorem -

①

Suppose that f is continuous on the closed interval $[a, b]$ and let N any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$ then there exists a number c in (a, b) such that $f(c) = N$



$\exists$ there exist

Example:- If $f(x) = x^2 + 5x - 1$ show that $\exists c \in (1, 2)$ such that $f(c) = 7$

① f cont on $[1, 2]$ Poly ② $f(1) = 5$ $f(2) = 13$ $f(a) \neq f(b)$
by I.V.T $5 < 7 < 13$ $\checkmark$ $\therefore \exists c \in (1, 2)$ such that $f(c) = 7$

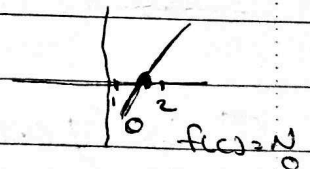
المركبة

Example: Show that there is root of the equation

$$4x^3 - 6x^2 + 3x = 2 \text{ between 1 and 2 ?}$$

$f(x)$, $[1, 2]$

$$4x^3 - 6x^2 + 3x - 2 = 0$$

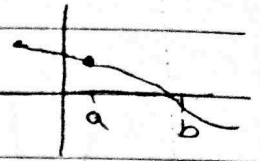


① f cont on $[1, 2]$ ② $f(1) = -1$ $f(2) = 12$ $-1 < 0 < 12$
then $\exists c \in (1, 2)$ such that $f(c) = 0$,

$\exists$ root between 1 and 2

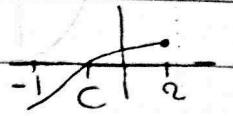
المركبة

Example: Show that the equation $x^3 - x - 1 = 0$ has at least one root.



$$f(x) = x^3 - x - 1 \quad f(-1) = -1 < 0 \quad f(0) = -1$$

$$f(1) = -1 \quad f(2) = 5 > 0$$



$$[-1, 2] \quad f(-1) < 0 < f(2) \rightarrow -1 < 0 < 5$$

① f cont $[-1, 2]$ ✓ كنه موجود

② $f(-1) \leq 0 \leq f(2) \quad -1 < 0 < 5$

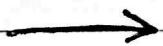
$\exists c \in [-1, 2]$ such that $f(c) = 0$

الاعتقاد ان اي فترات يكون لها حل (المعادن) صورة $\pm (1)$ or صورة $\pm (c)$

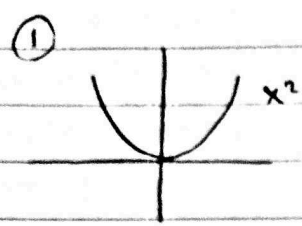
Lecture 41

limits at infinity

النهايات في اللانهاية



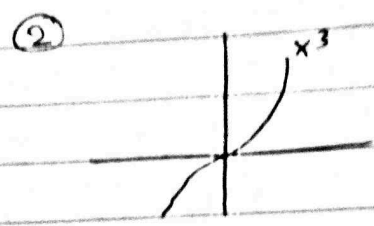
Lecture 41 limits at infinity



نحوه رسم نمودار

$$\lim_{x \rightarrow +\infty} x^2 = +\infty$$

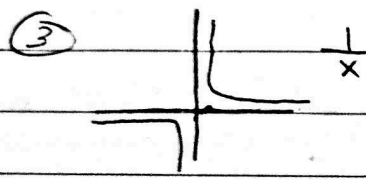
$$\lim_{x \rightarrow -\infty} x^2 = +\infty$$



$$\lim_{x \rightarrow +\infty} x^3 = +\infty$$

$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

ای نمودار
مربعی

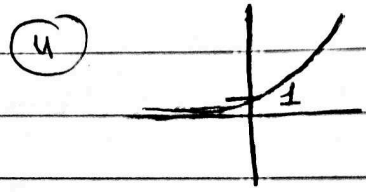


$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$



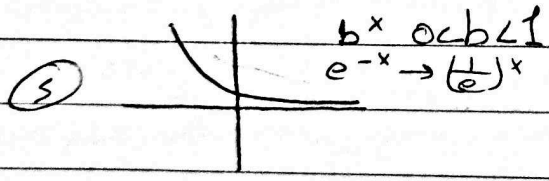
$$\lim_{x \rightarrow +\infty} b^x = +\infty$$

$$\lim_{x \rightarrow +\infty} e^x = +\infty$$

$$\lim_{x \rightarrow -\infty} b^x = 0$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

$b^x, b > 1$

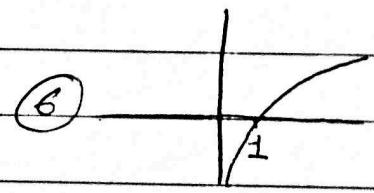


$$\lim_{x \rightarrow +\infty} e^{-x} = 0$$

$$\lim_{x \rightarrow +\infty} b^x = 0$$

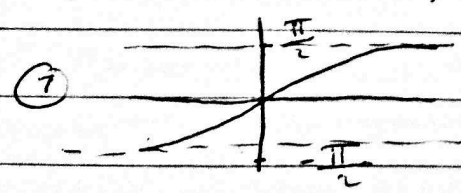
$$\lim_{x \rightarrow -\infty} e^{-x} = +\infty$$

$$\lim_{x \rightarrow -\infty} e^{-x} = +\infty$$



$$\lim_{x \rightarrow +\infty} \ln x = +\infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$



$\tan^{-1} x$

$$\lim_{x \rightarrow +\infty} \tan^{-1} x = +\frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Theorem:

$$\textcircled{1} \lim_{x \rightarrow \pm\infty} (f(x))^n = (\lim_{x \rightarrow \pm\infty} f(x))^n \quad n \text{ positive integer}$$

$$\textcircled{2} \lim_{x \rightarrow \pm\infty} k f(x) = k \lim_{x \rightarrow \pm\infty} f(x)$$

$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

$$\textcircled{3} \lim_{x \rightarrow \pm\infty} k = k$$

$$\lim_{x \rightarrow -\infty} x^2 = +\infty$$

$$\textcircled{4} \lim_{x \rightarrow +\infty} x^n = +\infty, \quad n = 1, 2, 3, 4, 5, \dots$$

$$-2(+\infty) = -\infty$$

$$\textcircled{5} \lim_{x \rightarrow -\infty} x^n \begin{cases} -\infty, & n = 1, 3, 5, \dots \text{ فرد} \\ +\infty, & n = 2, 4, 6, 8, \dots \text{ زوج} \end{cases}$$

$$-3(-\infty) = +\infty$$

في النهاية الموجبة والموجبة

$$\textcircled{6} \lim_{x \rightarrow \pm\infty} (a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

$$\textcircled{7} \text{ If } r > 0 \text{ is rational number then } \lim_{x \rightarrow \pm\infty} \frac{1}{x^r} = 0$$

$$-2(+\infty) = -\infty$$

$$-3(-\infty) = +\infty$$

$$a + \infty = +\infty$$

$$a - \infty = -\infty$$

Example

$$\textcircled{1} \lim_{x \rightarrow \infty} x^8 = +\infty$$

$$\textcircled{8} \lim_{x \rightarrow \infty} \left(\frac{314}{2.7}\right)^x = +\infty$$

$$\textcircled{2} \lim_{x \rightarrow -\infty} -4x^2 = -4 \lim_{x \rightarrow -\infty} x^2 = -\infty$$

$$\textcircled{9} \lim_{x \rightarrow -\infty} \frac{5}{x^4} = 5 \lim_{x \rightarrow -\infty} \frac{1}{x^4} = 5(0) = 0$$

$$\textcircled{3} \lim_{x \rightarrow -\infty} -3x^5 = -3 \lim_{x \rightarrow -\infty} x^5 = +\infty$$

$$\textcircled{9} \lim_{x \rightarrow \infty} e^{-\frac{2}{x}} = e^{\lim_{x \rightarrow \infty} \left(-\frac{2}{x}\right)} = e^{-2 \lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1$$

$$\textcircled{4} \lim_{x \rightarrow -\infty} (4x^2 - 5x + 4) = \lim_{x \rightarrow -\infty} -5x^6 = -5 \lim_{x \rightarrow -\infty} x^6 = -\infty$$

b^x sketch

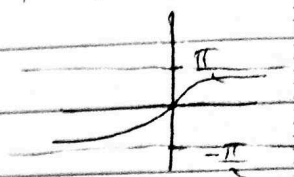
b^x $b > 1$

$$\textcircled{5} \lim_{x \rightarrow -\infty} 2^x = 0$$

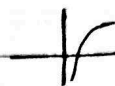
$$\textcircled{6} \lim_{x \rightarrow +\infty} 3^{-x} = \lim_{x \rightarrow +\infty} \left(\frac{1}{3}\right)^x = \lim_{x \rightarrow +\infty} \left(\frac{1}{3}\right)^x = 0$$

Lecture 42

$$\textcircled{1} \lim_{x \rightarrow \infty} \tan^{-1}(e^x) = \tan^{-1}\left(\lim_{x \rightarrow \infty} (e^x)\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$



$$\textcircled{2} \lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = \tan^{-1}\left(\lim_{x \rightarrow 0^+} \ln x\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$



$$\textcircled{3} \lim_{x \rightarrow -\infty} \frac{5-x^2}{3+2x^3+10x} = \lim_{x \rightarrow -\infty} \frac{-x^2}{2x^3} = \frac{-1}{2} \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

نأخذ الحد الأعلى
على الحد الأدنى

$$\textcircled{4} \lim_{x \rightarrow \infty} \frac{5+x^2-3x}{5+10x} = \lim_{x \rightarrow \infty} \frac{x^2}{10x} = \frac{1}{10} \lim_{x \rightarrow \infty} x = \infty$$

الاحتمالات كثيرة جداً
كثير جداً

① درجة البسط > درجة المقام
② درجة البسط = درجة المقام
③ درجة البسط < درجة المقام

مقابل أعلى من البسط على فاصلاته أو فاصلاته

زي ضال (3-4)

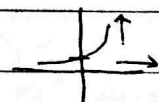
$$\textcircled{5} \lim_{x \rightarrow \infty} \frac{3+4x^5-2x^3}{4+x^5} = \lim_{x \rightarrow \infty} \frac{4x^5}{x^5} = \lim_{x \rightarrow \infty} 4 = 4$$

$$\textcircled{6} \lim_{x \rightarrow +\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left(\lim_{x \rightarrow +\infty} \frac{x+1}{x}\right) = \ln\left(\lim_{x \rightarrow +\infty} \frac{1}{1+\frac{1}{x}}\right) = \ln 1 = 0$$

في النهاية 0/0

$$\textcircled{7} \lim_{x \rightarrow \infty} \frac{1-e^x}{1+2e^x}$$

$$\text{let } y = e^x \quad x \rightarrow \infty, y \rightarrow \infty$$



$$\lim_{y \rightarrow \infty} \frac{1-y}{1+2y} = \lim_{y \rightarrow \infty} \frac{-y}{2y} = -\frac{1}{2}$$

$$\textcircled{8} \lim_{x \rightarrow -\infty} \frac{e^{2x} - e^{-x}}{e^{3x} - e^{-2x}} \quad y = e^x, x \rightarrow -\infty, y \rightarrow 0$$

$$\lim_{y \rightarrow 0} \frac{y^2 - y^{-1}}{y^3 - y^{-2}} = \lim_{y \rightarrow 0} \frac{\frac{y^2}{1} - \frac{1}{y}}{\frac{y^3}{1} - \frac{1}{y^2}} = \lim_{y \rightarrow 0} \left(\frac{y^3 - 1}{y} \right) \div \left(\frac{y^5 - 1}{y^2} \right)$$

$$\lim_{y \rightarrow 0} \left(\frac{y^3 - 1}{y} \cdot \frac{y^2}{y^5 - 1} \right) = \lim_{y \rightarrow 0} \frac{y(y^3 - 1)}{y^5 - 1} = \lim_{y \rightarrow 0} \frac{y^4 - y}{y^5 - 1} = \frac{0 - 0}{0 - 1} = 0$$

H.W. $\lim_{x \rightarrow \infty} \frac{e^{-x} - 3e^x}{e^{-1x} + 5e^x}$ $\begin{matrix} 0 & -\infty & \frac{-\infty}{+\infty} \end{matrix}$ let $y = e^x$
 $x \rightarrow \infty, y \rightarrow \infty$

$\frac{1}{b} >$
 $\frac{1}{a} < b < 1$

$$\lim_{y \rightarrow \infty} \frac{\frac{1}{y} - \frac{3y}{1}}{\frac{1}{y^2} + \frac{5y}{1}} \rightarrow \lim_{y \rightarrow \infty} \left(\frac{1 - 3y^2}{y} \right) \div \left(\frac{1 + 5y^3}{y^2} \right)$$

$$\lim_{y \rightarrow \infty} \left(\frac{1 - 3y^2}{y} \right) \cdot \left(\frac{y^2}{1 + 5y^3} \right) = \lim_{y \rightarrow \infty} \left(\frac{y - 3y^3}{1 + 5y^3} \right) = \frac{-3}{5} \neq$$

Lecture 43

Example

① $\lim_{x \rightarrow -\infty} \frac{\sqrt{5x^2 - 2}}{x + 3}$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(5 - \frac{2}{x^2})}}{x(1 + \frac{3}{x})}$$

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{5 - \frac{2}{x^2}}}{x(1 + \frac{3}{x})}$$

$$\lim_{x \rightarrow -\infty} \frac{|x| \sqrt{5 - \frac{2}{x^2}}}{x(1 + \frac{3}{x})}$$

$$= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{5 - \frac{2}{x^2}}}{x(1 + \frac{3}{x})} = \frac{-\sqrt{5 - 0}}{1 + 0} = -\sqrt{5}$$

$$\lim_{x \rightarrow \infty} \frac{2}{x^2} = 0$$

* Definition *

$$\rightarrow \sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\rightarrow \sqrt{x^4} = (\sqrt{x^2})^2 = (|x|)^2 = x^2$$

$$\rightarrow \sqrt{x^6} = (\sqrt{x^2})^3 = (|x|)^3 = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

$$\begin{array}{c} -x \quad x \\ \leftarrow \quad \rightarrow \\ 0 \end{array}$$

$$\lim_{x \rightarrow -\infty} |x| = \lim_{x \rightarrow -\infty} -x$$

$$\lim_{x \rightarrow \infty} |x| = \lim_{x \rightarrow \infty} x$$

$$\textcircled{2} \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^4 + x}}{x^2 - 8} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4} \sqrt{4 + \frac{1}{x^3}}}{x^2(1 - \frac{8}{x^2})} =$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 \sqrt{4 + \frac{1}{x^3}}}{x^2(1 - \frac{8}{x^2})} = \frac{\sqrt{4+0}}{1-0} = -\sqrt{4}$$

$$\textcircled{3} \lim_{x \rightarrow +\infty} (\sqrt{x^2 - 3x} - x) \left(\frac{\sqrt{x^2 - 3x} + x}{\sqrt{x^2 - 3x} + x} \right)$$

$$\lim_{x \rightarrow +\infty} \frac{(x^2 - 3x) - x^2}{\sqrt{x^2 - 3x} + x} = \lim_{x \rightarrow +\infty} \frac{-3x}{\sqrt{x^2} \sqrt{1 - \frac{3}{x}} + x} = \lim_{x \rightarrow +\infty} \frac{-3x}{x(\sqrt{1 - \frac{3}{x}} + 1)}$$

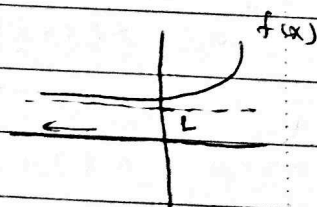
$$\frac{-3}{\sqrt{1} + 1} = \frac{-3}{2} \neq$$

$$\textcircled{\text{Hw}} \lim_{y \rightarrow +\infty} \frac{2-y}{\sqrt{7+6y^2}} \quad \lim_{y \rightarrow +\infty} \frac{y(\frac{2}{y} - 1)}{y\sqrt{7/y^2 + 6}} = \frac{0-1}{\sqrt{0+6}} = \frac{-1}{\sqrt{6}}$$

Definition

* The line $y=L$ is called horizontal asymptote of the curve $y=f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$

Example: Find horizontal asymptote for



$$\textcircled{1} f(x) = \frac{\sqrt{x^2 + 7} - 4x}{x+3} \quad \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2 + 7} - 4x}{x+3} = \lim_{x \rightarrow +\infty} \frac{x\sqrt{1 + \frac{7}{x^2}} - 4x}{x(\frac{1}{x} + \frac{3}{x})}$$

$$\lim_{x \rightarrow +\infty} \frac{x(\sqrt{1 + \frac{7}{x^2}} - 4)}{x(1 + \frac{3}{x})} = \frac{\sqrt{1+0} - 4}{1+0} = -3 \neq$$

at $y=3$ horizontal asymptote

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+7} - 4x}{x+3} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{1+7/x^2} - 4x}{x(1+3/x)}$$

$$\lim_{x \rightarrow -\infty} \frac{x(-\sqrt{1+7/x^2} - 4)}{x(1+3/x)} = \frac{-1-4}{1} = -5 \neq$$

at $y=5$ horizontal asymptote

(2) $f(x) = e^x$

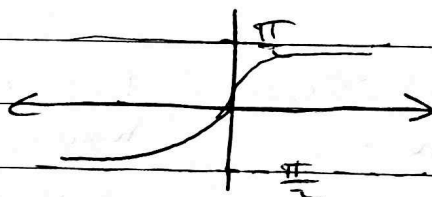
$$\lim_{x \rightarrow \infty} e^x = \infty, \quad \lim_{x \rightarrow -\infty} e^x = 0$$



at $y=0$ horizontal asymptote

(3) $f(x) = \tan^{-1} x$

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$



$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

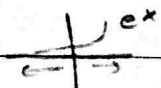
at $y = \frac{\pi}{2}, -\frac{\pi}{2}$ horizontal asymptote

- $\frac{0}{0}, \frac{\infty}{\infty} \rightarrow 0$
- $\frac{\infty}{\infty} \rightarrow \infty$
- $\frac{\pm \infty}{0} \rightarrow \pm \infty$
- $\frac{\infty}{\infty} \rightarrow \infty$
- $\frac{+\infty + \infty}{+\infty} \rightarrow +\infty$
- $\frac{+\infty - \infty}{+\infty} \rightarrow -\infty$

① $\frac{0}{0}, \frac{\pm \infty}{\pm \infty}$

② $+\infty, -\infty, -\infty + \infty, 0 \cdot \pm \infty$

③ $\frac{\pm \infty}{\pm \infty}, \frac{\pm \infty}{0}, 0^0$



$$\frac{\pm \infty}{\pm \infty} = \frac{\pm \infty}{\pm \infty}$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x} = \frac{\infty}{\infty}$$

$$\frac{+\infty}{+\infty} = \frac{+\infty}{+\infty}$$

$$\frac{-\infty}{-\infty} = \frac{-\infty}{-\infty}$$

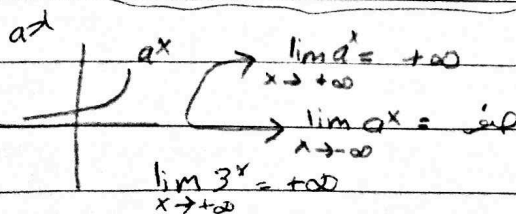
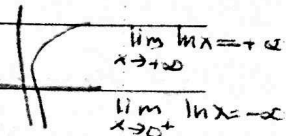
$$\frac{\pm \infty}{\pm \infty} = \frac{\pm \infty}{\pm \infty}$$

محدد القوسية

محدد القوسية

$$\lim_{x \rightarrow \infty} e^x = +\infty, \quad \lim_{x \rightarrow -\infty} e^x = 0$$

$$\ln(+\infty) = +\infty, \quad \ln(0^+) = -\infty$$



$$\lim_{x \rightarrow +\infty} 3^x = +\infty$$

$$\tan^{-1}(-\infty) = -\frac{\pi}{2}, \quad \tan^{-1}(+\infty) = \frac{\pi}{2}$$



$$\lim_{x \rightarrow -\infty} a^x = 0, \quad \lim_{x \rightarrow +\infty} a^x = +\infty$$



Five Apple

Lecture 44

Derivatives

الاشتقاق

Definition:- The derivative of a function f at a number a denoted

$f'(a)$ is

$y'(a)$

$\frac{dy}{dx} \frac{dF}{dx} \bigg|_{x=a}$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

The derivative of f with respect to x

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

$$f'(x) = \frac{dy}{dx}$$

Example:- $f(x) = x^2$ Find $f'(x)$, $f'(2)$?

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = 2x$$

$$f'(2) = 2(2) = 4$$

$$\bullet f(x) = c \rightarrow f'(x) = 0$$

$\downarrow$
c constant

$$\text{Example:- } \textcircled{1} f(x) = 3 \rightarrow f'(x) = 0$$

$$\textcircled{2} f(x) = x^3 \rightarrow f'(x) = 0$$

$$\bullet f(x) = x \rightarrow f'(x) = 1$$

$$\bullet f(x) = x^r \rightarrow f'(x) = r x^{r-1}$$

Example: ① $f(x) = \sqrt{x} = x^{1/2} \rightarrow f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

② $f(x) = x^7 \rightarrow f'(x) = 7x^6 \rightarrow f'(1) = 7 \leftarrow y'(1) \rightarrow \frac{dy}{dx} \Big|_{x=1}$

④ $\frac{d}{dx}(cf(x)) = cf'(x) \quad , \quad \frac{d}{dx}\left(\frac{f(x)}{c}\right) = \frac{1}{c} f'(x)$

Example: ① $f(x) = 3x^4 = 12x^3$

② $f(x) = \frac{x^6}{5} = \frac{1}{5}(6)x^5$

③ $(f \pm g)' = f' \pm g' \rightarrow f(x) = 3x^7 - 2x^4 + x$ Find $f'(x) \rightarrow 21x^6 - 8x^3 + 1$

④ $(f \cdot g)' = f' \cdot g + f \cdot g' \rightarrow (x^2+1)(x^3-2x)$ Find $\frac{dy}{dx} \Big|_{x=1}$

$y' = (2x) \cdot (x^3-2x) + (x^2+1) \cdot (3x^2-2)$

$\frac{dy}{dx} \Big|_{x=1} = (2) \cdot (-1) + (2) \cdot 1 = -2 + 2 = 0 \#$

⑦ $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$ Example: $y = \frac{3x+4}{x^2+1}$ Find $y'(1)$?

$y' = \frac{(3) \cdot (x^2+1) - (2x) \cdot (3x+4)}{(x^2+1)^2} = \frac{dy}{dx} \Big|_{x=1} = \frac{(3)(2) - (2)(7)}{4} = \frac{6-14}{4} = \frac{-8}{4} = -2 \#$

② $f(2) = -3$, $f'(2) = 4$, $g(2) = 1$, $g'(2) = 5$ Find

① $\left(\frac{3f}{g}\right)'(2) = \frac{3f'(2)g(2) - g'(2)(3f(2))}{(g(2))^2} = \frac{3(4)(1) - (5)(3(-3))}{1}$

$= \frac{12 + 45}{1} = 57$ or $= 3 \left(\frac{f'(2)g(2) - g'(2)f(2)}{g(2)} \right) = 3 \left(\frac{(4)(1) - (5)(-3)}{1} \right) = 3(4 + 15) = 57$

⑥ $\left(\frac{3f(2)}{g(2)}\right)' = \left(\frac{3f(2)}{g(2)}\right)' = \left(\frac{(3)(-3)}{1}\right)' = (-9)' = 0$

⑨ دایره ای که مرکز آن ثابت باشد و شعاع آن متغیر باشد

⑧ $\left(\frac{c}{f}\right)' = -\frac{c \cdot f'}{f^2}$ $c = \text{constant}$

Lecture 45

Example: $\frac{2}{3x+1} = \frac{-2(3)}{(3x+1)^2} = \frac{-6}{(3x+1)^2}$

② $\frac{d}{dx} \left(x^2 \frac{d}{dx} \left(\frac{1}{x} \right) \right) \rightarrow \frac{d}{dx} \left(x^2 \cdot \frac{-1}{x^2} \right) = 0$

⑨ f f'

$\sin x$ $\cos x$

$\cos x$ $-\sin x$

$\tan x$ $\sec^2 x$

$\cot x$ $-\csc^2 x$

$\sec x$ $\sec x \tan x$

$\csc x$ $-\csc x \cot x$

Example:

$f(x) = \tan x + 3x^2 \cos x - \frac{5x \cos x}{\sec x}$ Find $f'(x)$

$\sec^2 x + (6x \cos x + (3x^2)(-\sin x)) - (5)(\cos x) + \sin x$

$\sec^2 x + 18x^3 (\cos x) - 5 \cos x + 5x \sin x$

$5x \cos x$

$\sec^2 x + (3x^2)(-\sin x) + \cos x (6x) + (5x(-\sin x) + \cos x 5)$

⑩ chain rule:

$$y = z = x$$

If $y = f(z)$ then $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$
 $z = g(x)$

Examples:-

① If $y = 3z^3 + 2$, $z = x^2 + 4$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

لفرض كان $z = x$

$$\frac{dy}{dz} = 6z^2 \quad \frac{dz}{dx} = 2x \quad \frac{dy}{dx} = 6z^2 \cdot 2x$$

$$\frac{dy}{dx} = 6(x^2 + 4) \cdot 2x = 12x(x^2 + 4)^2$$

② If $x = \cos n$, $\frac{dn}{dm} = 3$ Find $\frac{dx}{dm}$ when $n = \frac{3\pi}{2}$ $x = n = m$

$$\frac{dx}{dm} = \frac{dx}{dn} \cdot \frac{dn}{dm} = 3$$

$$\frac{dx}{dn} = -\sin n \rightarrow \frac{dx}{dm} = -3\sin(n)$$

$$= -3\sin\left(\frac{3\pi}{2}\right) = +3$$

$$\begin{array}{c} (0, +1) \\ | \\ (0, -1) \\ \hline \frac{2\pi}{3} \end{array}$$

lecture 1/6

$$(1) (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example: $f(x) = x^2$ $g(x) = 3x+1$ Find $(g \circ f)'(x)$

$$g'(f(x)) f'(x) = g'(x^2) \cdot 2x = 6x$$

$$(2) (fg)' = f'(g(x)) g'(x) = f'(3x+1)(3) = (6x+2)3 = 18x+6$$

(2) $f(9) = 5$, $g(2) = 9$, $g'(2) = -3$, Find

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2) = f'(9)(-3) = 5(-3) = -15$$

(3) If $f(g(x)) = 3x^2 + 8x$ Find $F'(2)$

$$f'(2x) \cdot g'(x) = 6x + 8 \rightarrow f'(2x) = 3x + 4$$

$2x = 2$ $x = 1$ $f'(2) = 3 + 4 = 7$

(4) If $\frac{d}{dx}(f(3x)) = 6x$ Find $\frac{d}{dx}(f(x))$

$$f'(3x) \cdot 3 = 6x$$

$$f'(3x) = 2x$$

$$f'(y) = \frac{2}{3}y \rightarrow f'(x) = \frac{2}{3}x$$

$$(12) \frac{d}{dx} (f(x)^n) = n(f(x)^{n-1}) \cdot f'(x)$$

Example:-

$$(1) f(x) = (x^3 + 5)^{27} \text{ find } f'(x)$$

$$f'(x) = 27(x^3 + 5)^{26} \cdot (3x^2) \rightarrow (3)(27) x^2 (x^3 + 5)^{26}$$

$$(2) y = f^5(3x + 4) \text{ find } \frac{dy}{dx}$$

$$F^5 = (f(x))^5 \text{ يعني امواس}$$

المشتقة الخامسة = $f^{(5)}$

المشتقة الخامسة $f^{(5)}$ بين قوسين

$$y = (f(3x + 4))^5$$

$$\frac{dy}{dx} = 5(f(3x + 4))^4 \cdot f'(3x + 4) \cdot 3 = 15 f^4(3x + 4) \cdot f'(3x + 4)$$

$$(3) y = \csc^3 x \text{ Find } y'$$

$$\sin^2 x = (\sin x)^2$$

$$y = (\csc x)^3 \rightarrow \frac{dy}{dx} = 3(\csc x)^2 \cdot (-\csc x \cot x)$$

$$\csc^5 x = (\csc x)^5$$

$$\frac{dy}{dx} = -3 \csc^3 x \cot x$$

(13)

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(f(x))$	$f'(x) \cos(f(x))$	$\cot(f(x))$	$-f'(x) \csc^2(f(x))$
$\cos(f(x))$	$-f'(x) \sin(f(x))$	$\sec(f(x))$	$f'(x) \sec(f(x)) \tan(f(x))$
$\tan(f(x))$	$f'(x) \sec^2(f(x))$	$\csc(f(x))$	$-f'(x) \csc(f(x)) \cot(f(x))$

Example:- Find $\frac{dy}{dx}$

$$(1) f(x) = \cot(x^2) \rightarrow f'(x) = -\csc^2(x^2) \cdot 2x = -2x \csc^2(x^2)$$

$$\textcircled{2} y = \cos^2(3x^5) \rightarrow (\cos(3x^5))^2 =$$

$$2(\cos(3x^5)) \cdot (-\sin(3x^5)) \cdot 15x^4$$

$$= -30 \sin(3x^5) \cos(3x^5)$$

$$\sin(2a) = 2 \sin a \cos a$$

$$= -15 \sin(6x^5) \quad \text{Ans}$$

$$\textcircled{3} y = \sin(\cos(\tan x))$$

$$\cos(\cos(\tan x)) (-\sin(\tan x)) \sec^2 x$$

$$= -\sec^2 x \sin(\tan x) \cos(\cos(\tan x))$$

$$\textcircled{14} \frac{d}{dx} (\sqrt{f(x)}) = \frac{f'(x)}{2\sqrt{f(x)}}$$

Example:-

$$\textcircled{1} y = \sqrt{\tan x} \quad \frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

$$2x + y + xy' = 0$$

lecture 47

15 Implicit Differentiation

$$y = x^2 + x + 5 \text{ explicit}$$

$$yx + y^2 + 1 = x \text{ implicit}$$

مباشرة

خفية

متغير بدلالة متغير
خلاف

$$\frac{d}{dx}(x) = 1 \cdot \frac{dx}{dx} = 1$$

$$\frac{d}{dy}(y) = 1 \cdot \frac{dy}{dy} = y'$$

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx} = 2yy'$$

$$\frac{d}{dx}(x^2 + y^2) = 2x \frac{dx}{dx} + 2y \frac{dy}{dx} = 2x + 2y \frac{dy}{dx} = 2x + 2yy'$$

$$\frac{d}{dx}(xy) = 1 \cdot \frac{dx}{dx} y + x \cdot 1 \cdot \frac{dy}{dx} = y + x \frac{dy}{dx} = y + xy'$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{1 \cdot \frac{dx}{dx} \cdot y - x \cdot 1 \cdot \frac{dy}{dx}}{y^2} = \frac{y - xy'}{y^2}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1 \cdot \frac{dy}{dx} \cdot x - 1 \cdot y}{x^2} = \frac{y'x - y}{x^2}$$

x2

Example:-

$$\textcircled{1} x^2 + xy = 1 \text{ Find } \frac{dy}{dx}$$

$$2x + y + x \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-2x - y}{x} = -2 - \frac{y}{x}$$

$$\textcircled{2} x^2 + xy = 1 \text{ Find } \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = 0 - \left(\frac{\frac{dy}{dx} x - y}{x^2} \right) = - \left(\frac{-2x - y}{x} \cdot \frac{x - y}{x^2} \right)$$

$\frac{dy}{dx}, f', y' \leftarrow$ الأولى
 $\frac{d^2y}{dx^2}, y'', f'' \leftarrow$ الثانية
 $y''', f''' \leftarrow$ الثالثة

$$\frac{2x + y + y}{x^2} = \frac{2x + 2y}{x^2}$$

③ If $\sin(xy^2) + 3xy = 1 - y$ Find $\frac{dy}{dx}$ when $x=0$

$$\text{When } x=0 \rightarrow \sin(0 \cdot y^2) + 3 \cdot 0 \cdot y = 1 - y \rightarrow 0 + 0 = 1 - y \rightarrow \boxed{y=1}$$

$$\left. \frac{dy}{dx} \right|_{\substack{x=0 \\ y=1}} = ? \rightarrow (y^2 + 2xyy') \cos(xy^2) + 3y + 3xy' = 0 - y'$$

$$y'(0,1) = ? \quad (1^2 + 0(2)(1)y'_{(0,1)}) \cos(0(1)^2) + 3(1) + 3(0)y'_{(0,1)} = -y'_{(0,1)}$$

$$1 + 3 + 0 = -y'_{(0,1)} \rightarrow -y'_{(0,1)} = -4$$

$$\left. \frac{dy}{dx} \right|_{\substack{x=0 \\ y=1}} = -4$$

④ If $x^2 + y^2 = 9$ show that $\frac{d^2y}{dx^2} = \frac{-9}{y^3}$?

$$2x + 2yy' = 0 \rightarrow y' = \frac{-x}{y} \rightarrow \frac{dy}{dx} = \frac{-x}{y} \quad \text{b.s.c}$$

$$\frac{d^2y}{dx^2} = \frac{-y - (-xy')}{y^2} = \frac{-y + x(\frac{-x}{y})}{y^2} = \frac{d^2y}{dx^2} = \frac{-y + \frac{-x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = - \frac{y^2 + x^2}{y^3} = \frac{-x^2 - y^2}{y^3} = \frac{-9}{y^3} \rightarrow \boxed{x^2 + y^2 = 9}$$

lecture 48

$$(16) f(x) = \log_a(g(x)) \rightarrow f'(x) = \frac{g'(x)}{g(x) \cdot \ln a}$$

$$f(x) = \ln(g(x)) \rightarrow f'(x) = \frac{g'(x)}{g(x) \ln e}$$

Example:-

$$(1) y = \log_{10}(x^2+1) \rightarrow \frac{dy}{dx} = \frac{2x}{(x^2+1) \ln 10}$$

$$(2) y = \log_7(\cot x) \rightarrow \frac{dy}{dx} = \frac{-\csc^2 x}{\cot x \cdot \ln 7}$$

$$(3) y = (\ln x)^3 \rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{1}{x} = \frac{3(\ln x)^2}{x}$$

$$(4) y = \cos(\ln x) \rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot -\sin x(\ln x) = -\frac{\sin \ln x}{x}$$

نستخدم قوانين الـ ln

$$(5) y = \ln\left(\frac{x^2 \sin x}{x+1}\right) \rightarrow \ln(x^2 \sin x) - \ln(x+1) \rightarrow \ln(x^2) + \ln(\sin x) - \ln(x+1)$$

$$\frac{dy}{dx} = \frac{2x}{x^2} + \frac{\cos x}{\sin x} - \frac{1}{x+1} = \frac{2}{x} + \cot x - \frac{1}{x+1}$$

$$(6) y = \log_x 2 \rightarrow \log_a b = \frac{\ln a}{\ln b} \rightarrow \frac{\ln 2}{\ln x} = \frac{-\ln 2 \cdot \frac{1}{x}}{(\ln x)^2}$$

$$\frac{dy}{dx} = \frac{-\ln 2}{x(\ln x)^2}$$

مشتق

NOT

$$(7) y = (x^2+1)^{\sin x} = \ln y = \sin x \ln(x^2+1)$$

$$y \left(\frac{y'}{y} \right) = \cos x \cdot \ln(x^2+1) + \sin x \left(\frac{2x}{x^2+1} \right)$$

$$\frac{dy}{dx} = \left(\cos x \cdot \ln(x^2+1) + \sin x \left(\frac{2x}{x^2+1} \right) \right) y \rightarrow (x^2+1)^{\sin x}$$

* (عدد) $\leftarrow$ م $\leftarrow$ القاعدة القوية
 * (اقتران) $\leftarrow$ عدد (اقتران) $\leftarrow$ قاعدة
 * (اقتران) $\leftarrow$ ناتج $\leftarrow$ في التفاضل

$$\frac{g'}{1+g'}$$

$$\frac{g'}{2\sqrt{g^2+1}}$$

$$\frac{g'}{\sqrt{1+g^2}}$$

$$(17) f(x) = a^{g(x)} \rightarrow f'(x) = g'(x) \cdot a^{g(x)} \cdot \ln a$$

$$f(x) = e^{g(x)} \rightarrow f'(x) = g'(x) \cdot e^{g(x)} \cdot 1/e$$

Example: ① $y = e^{3x} \quad \frac{dy}{dx} = 3e^{3x}$

② $y = 2^{\sin x} \quad \frac{dy}{dx} = \cos x \cdot 2^{\sin x} \cdot \ln 2$

③ $y = \ln(\cos(e^x)) \quad \frac{dy}{dx} = \frac{\sin(e^x) \cdot e^x \cdot -1}{\cos(e^x)} = -e^x \tan(e^x)$

④ $y = 2^{(3^{x^2})} = g(x) \quad \frac{dy}{dx} = g'(x) \cdot 2^{3^{x^2}} \cdot \ln 2$

$$g(x) = 3^{x^2}$$

$$g'(x) = 2x3^{x^2} \ln 3$$

$$2x3^{x^2} \ln 3 \cdot 2^{3^{x^2}} \cdot \ln 2$$

lecture 49

(19)

f

f'

$$\frac{1}{\sqrt{1-x^2}}$$

$$\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{1}{1+x^2}$$

$$\frac{-1}{1+x^2}$$

$$\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{-1}{x\sqrt{x^2-1}}$$

$$\sin^{-1}(g(x))$$

$$\frac{g'(x)}{\sqrt{1-(g(x))^2}}$$

inverse of قواعد

$$\cos^{-1}(g(x))$$

$$\frac{-g'(x)}{\sqrt{1-g^2}}$$

$$\tan^{-1}(g(x))$$

$$\frac{g'}{1+g^2}$$

$$\sin \frac{g'}{\sqrt{1-g^2}}$$

$$\tan \frac{g'}{1+g^2}$$

$$\cot^{-1}(g(x))$$

$$\frac{-g'}{1+g^2}$$

$$\sec \frac{g'}{g\sqrt{g^2-1}}$$

$$\sec^{-1}(g(x))$$

$$\frac{g'}{g\sqrt{g^2-1}}$$

$$\frac{g'}{\sqrt{1-g^2}}$$

$$\csc^{-1}(g(x))$$

$$\frac{-g'}{g\sqrt{g^2-1}}$$

$$\frac{g'}{g^2+1}$$

$$\frac{g'}{g\sqrt{g^2-1}}$$

Example:

$$① y = \sin^{-1}(5x) \rightarrow \frac{dy}{dx} = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$$

$$② y = \tan^{-1}(3x^2) \rightarrow \frac{dy}{dx} = \frac{6x}{1+(3x^2)^2} = \frac{6x}{1+9x^4}$$

$$③ y = \log(\cos^{-1}x) \rightarrow \frac{dy}{dx} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\cos^{-1}(x) \cdot \ln 10}$$

$$= \frac{-1}{\ln 10 \cdot \cos^{-1}(x) \cdot \sqrt{1-x^2}}$$

$$④ y = 2^{\cot^{-1}x} \quad \frac{dy}{dx} = \frac{-1}{1+x^2} 2^{\cot^{-1}x} \cdot \ln 2 = \frac{-\ln 2 \cdot 2^{\cot^{-1}x}}{1+x^2}$$

المشتقات العالية

* Higher derivatives $f', f'', f''', f^{(4)}, f^{(5)}, f^{(6)}, \dots$ $f^{(4)} = (f''')'$
 $f^{(4)}$ ← المشتقة الرابعة

Example:- ① $f(x) = 5x^7 + 2x^5 + 3x \rightarrow f'(x) = 5(7)x^6 + 2(5)x^4 + 3$

$$f'' = (5(7)(6)x^5 + 2(5)(4)x^3 + 0 \rightarrow f''(x) = (5(7)(6)(5)x^4 + (2(5)(4)(3)x^2$$

$$f^{(4)} = (5(7)(6)(5)(4)x^3 + (2(5)(4)(3)(2)x$$

② $f(x) = 2^x$ Find $f^{(100)}(x) \rightarrow f'(x) = 2^x \cdot \ln 2 \rightarrow f''(x) = \ln 2 \cdot 2^x \cdot \ln 2$
 $= (\ln 2)^2 \cdot 2^x \rightarrow$

$$f'''(x) = (\ln 2)^2 \cdot 2^x \cdot \ln 2 = (\ln 2)^3 2^x$$

$$f^{(4)}(x) = (\ln 2)^3 2^x \ln 2 = (\ln 2)^4 2^x$$

$$f^{(5)}(x) = (\ln 2)^5 2^x$$

$$f^{(6)}(x) = (\ln 2)^6 2^x$$

$$f^{(100)}(x) = (\ln 2)^{100} 2^x$$

$$f^{(n)}(x) = (\ln 2)^n 2^x$$

Lecture 50

Differentiability:-

قابل الاشتقاق

موجوده

Theorem - The function f is differentiable at $x=a$ if $f'(a)$ exist

Theorem -

↔

① If f is differentiable at $x=a$ then f is continuous at $x=a$

② If f is not continuous at $x=a$ then f is not differentiable at $x=a$

***Example :-**

① $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ 2x, & x > 1 \end{cases}$ is f differentiable at $x=1$?

* اذا غير متصل فهو غير قابل الاشتقاق

① $f(1) = 2$ ② $\lim_{x \rightarrow 1^+} 2x = 2$ ③ $\lim_{x \rightarrow 1^-} (x^2+1) = 2$ ④ $f(1) = \lim_{x \rightarrow 1} f = 2$ ✓

$f'(x) = \begin{cases} 2x, & x < 1 \\ 2, & x > 1 \end{cases}$

$x=1$

$f'_+(1) = 2, f'_-(1) = 2$

$f'_+(1) = f'_-(1) = 2 \therefore f'(1) = 2$ exist

② $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ x^3+1, & x > 1 \end{cases}$

is f differentiable at $x=1$?

① $f(1) = 2$ ② $\lim_{x \rightarrow 1^+} (x^3+1) = 2 = \lim_{x \rightarrow 1^-} (x^2+1) = 2$ ③ $\lim_{x \rightarrow 1} f = f(1)$

$\therefore f$ cont at $x=1$

$f'_+(1) = 3(1)^2 = 3, f'_-(1) = (2)(1) = 2$

$f'_+(1) \neq f'_-(1), f'(1)$ d.n.e

$f'(x) = \begin{cases} 2x, & x < 1 \\ 3x^2, & x > 1 \end{cases}$ $\therefore f$ not differentiable at $x=1$

③ $f(x) = \begin{cases} x^2+1, & x > 1 \\ 2x+5, & x \leq 1 \end{cases}$ is f differentiable at $x=1$

④ $f(1) = 7$ $\ominus \lim_{x \rightarrow 1^+} (x^2+1) = 2$, $\neq \lim_{x \rightarrow 1^-} (2x+5) = 7$ $\lim f$ d.n.e

$\therefore f$ not cont at $x=1$ $\therefore$ by theorem f not differentiable at $x=1$

④ $f(x) = \begin{cases} 3x^2, & x \leq 1 \\ ax+b, & x > 1 \end{cases}$ is differentiable at $x=1$ find a and b

* f differentiable at $x=1$, $f'(1)$ exist

* f differentiable at $x=1$ then f cont at $x=1$

$f(x) = \begin{cases} 6x, & x \leq 1 \\ a, & x > 1 \end{cases}$ f differentiable $f'_+(1) = f'_-(1)$

$a = 6(1)$ $\boxed{a=6}$ $\therefore f$ differentiable $\therefore f$ cont at $x=1$

$f(1) = \lim_{x \rightarrow 1^+} f(x) \rightarrow 3(1)^2 = \lim_{x \rightarrow 1^+} (ax+b)$

$3 = 6(1) + b$, $3 - 6 = b$ $\boxed{b=-3}$

Example:-

① $f(4) = 3$ $f'(4) = 9$ find

$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$\lim_{h \rightarrow 0} \frac{f(4+h) - 3}{5h}$

$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

$\frac{1}{5} \lim_{h \rightarrow 0} \frac{f(4+h) - 3}{h} = \frac{1}{5} \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \frac{1}{5} f'(4) = \frac{1}{5} (9) = \frac{9}{5}$

$f'(a)$

If $\lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = 2$ then $f(x)$ may equal to

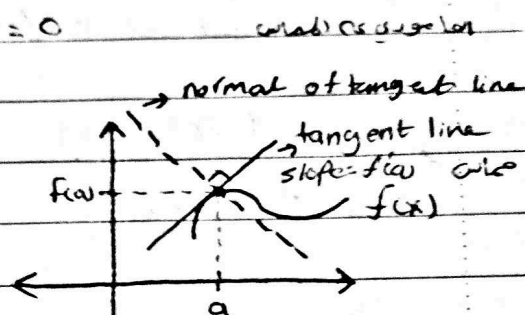
(a) $2\sqrt{x} \rightarrow \frac{2\sqrt{4}}{2\sqrt{4}} = 2$ If $f'(4) = 2$ then $f(x) =$

(b) $-8\sqrt{x} \rightarrow \frac{-8}{2\sqrt{4}} = -2$

(c) $2\cos(x-4) \rightarrow -2\sin(x-4) \rightarrow -2\sin(0) = 0$

(d) $x^2 - 4 \rightarrow 2x \rightarrow (2)(4) = 8$

Lecture 51



* Equation of tangent line at $x = a$ $f'(a) \cdot \frac{-1}{f'(a)} = -1$

- point of tangency $(a, f(a))$
- slope of tangent line $= f'(a) = \frac{dy}{dx} \Big|_{(a, f(a))}$
- equation of tangent line: $y - f(a) = f'(a)(x - a)$

• slope of normal (perpendicular) $= \frac{-1}{f'(a)} = \frac{-1}{\text{slope of tangent line}}$

• equation of normal: $y - f(a) = \frac{-1}{f'(a)}(x - a)$

Remarks:- slope = slope متوازي

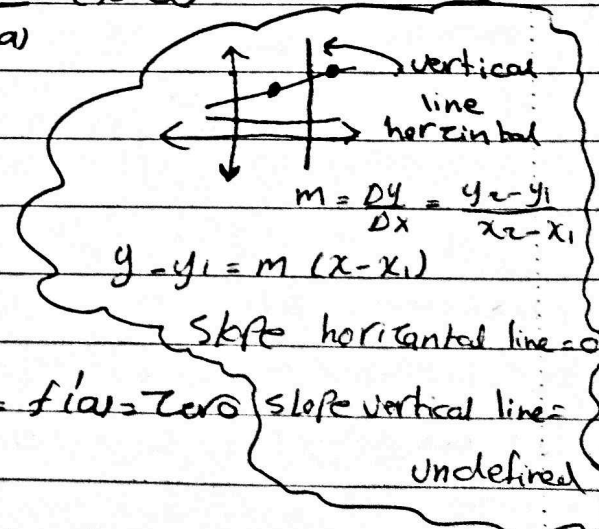
① If $l_1 \parallel l_2 \rightarrow m_1 = m_2$ (Parallel //)

② If $l_1 \perp l_2 \rightarrow m_1 \times m_2 = -1$

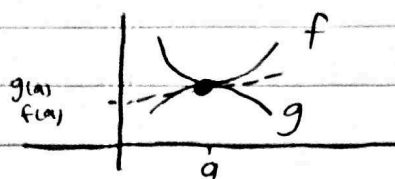
③ The slope of horizontal line = zero

The slope of horizontal tangent line $= f'(a) = \text{zero}$ slope vertical line =

$\frac{1}{k}$ $\frac{1}{a}$ $f'(a) = 0$ $\frac{1}{\text{slope}}$



④ $f(a) = g(a)$
 $f'(a) = g'(a)$



Example:-

① Find the equation of tangent line for
 $y = x \tan^{-1} x$ at $x=1$?

* Point of tangency = $(1, f(1)) = (1, 1 \tan^{-1} 1) = (1, \frac{\pi}{4})$

* Slope of tangent line = $f'(x) = \tan^{-1} x + x \cdot \frac{1}{1+x^2} = \tan^{-1} x + \frac{x}{1+x^2}$

* Slope of tangent line: $f'(1) = \tan^{-1}(1) + \frac{1}{1+1^2} = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi+2}{4}$

* Equation of tangent line: $y - f(a) = f'(a)(x - a)$

$y - \frac{\pi}{4} = \frac{\pi+2}{4}(x-1) \Rightarrow y = -$

* Equation of normal $\rightarrow y - f(a) = \frac{-1}{f'(a)}(x - a)$

$y - \frac{\pi}{4} = \frac{-1}{\frac{\pi+2}{4}}(x-1) \rightarrow y - \frac{\pi}{4} = \frac{-4}{\pi+2}(x-1)$

② Find the slope of tangent line and slope of normal

$2x^2 + 3e^{x+1} + 3xy + y^2 = 9$ at $(-1, 1)$

$4x + 3e^{x+1} + 3y + 3xy' + 2yy' = 0$

$4(-1) + 3e^{-1+1} + 3(1) + 3(-1)y' + 2(1)y' = 0$

slope of tangent = $y'_{(-1,1)}$

slope of Normal = $\frac{-1}{y'_{(-1,1)}}$

$-4 + 3 + 3 - 3y'_{(-1,1)} + 2y'_{(-1,1)} = 0 \rightarrow 2 - y'_{(-1,1)} = 0$

* Slope of tangent = $y'_{(-1,1)} = 2$

* Slope of Normal = $\frac{-1}{y'_{(-1,1)}} = \frac{-1}{2}$

Equation of tangent $\rightarrow (y-1) = 2(x+1)$

Normal $\rightarrow (y-1) = \frac{-1}{2}(x+1)$

$y'_{(-1,1)} = 2$

③ Find all values of x at which the tangent line of the curve

اوجد قيم x التي يكون فيها مماس

$f'(a)$: slope
slope = 0

① $y = x^5 e^x$ (horizontal) $\rightarrow$ is parallel



$$y' = 5x^4 e^x + x^5 e^x = 0$$

$$x^4 e^x (5 + x) = 0$$

$$x^4 = 0 \rightarrow x = 0$$

$$e^x \neq 0 \rightarrow e^x \rightarrow \neq 0 \quad (0, 100) \text{ value}$$

$$5 + x = 0 \rightarrow x = -5$$

at $x = 0$, $x = -5$ horizontal tangent line

Value of y

points $(0, 0)$

Point (-5)

$$(-5, (-5)^5 e^{-5})$$

$$f'(0) = 0$$

$$f'(-5) = 0$$

H.W

② $y = \frac{x^2 - 1}{x + 2}$ horizontal

$$y' = \frac{(2x)(x+2) - (1)(x^2-1)}{(x+2)^2} = \frac{2x^2 + 4x - x^2 + 1}{(x+2)^2} = 0$$

$$\frac{x^2 + 4x + 1}{(x+2)^2} = 0$$

$$x^2 + 4x + 1 = 0$$

$$a = 1 \quad b = 4 \quad c = 1$$

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$$

$$\rightarrow \frac{-4 \pm \sqrt{12}}{2}$$

$$\rightarrow \frac{-4 \pm 2\sqrt{3}}{2}$$

$$x = -2 \pm \sqrt{3}$$

How

④ Find all points at which the tangent line of the curve $x^2 + 2x + y^4 + 4y = 5$ vertical?

$$2x + 2 + 4y^3 y' + 4y' = 0 \rightarrow y' \frac{(4y^3 + 4)}{4y^3 + 4} = \frac{-2x - 2}{(4y^3 + 4)}$$

$$y' = \frac{-2x - 2}{(4y^3 + 4)} \rightarrow 4y^3 + 4 = 0 \rightarrow 4(y^3 + 1) \rightarrow y = -1$$

for $y = -1$ $x^2 + 2x - 8 = (x+4)(x-2)$ $x = -4, x = 2$

points $\rightarrow (-4, -1), (2, -1)$

⑤ Find the equation of tangent line of $y = f(a) = f'(a)(x - a)$

① $y = \tan x$ at $x = \frac{\pi}{4}$ $(\frac{\pi}{4}, 1)$

② $y = x \cos 3x$ at $x = \pi$ $(\pi, -\pi)$

③ $x^2 + y^2 = 5$ at $(\sqrt{5}, 0)$ $(\sqrt{5}, 0)$

$$y - 1 = f'(a)(x - \frac{\pi}{4}) \Rightarrow y - 1 = 2(x - \frac{\pi}{4})$$

$$y + \pi = f'(a)(x - \pi) \Rightarrow y + \pi = -1(x - \pi)$$

$$y = f'(a)(x - \sqrt{5}) \Rightarrow x = \sqrt{5}$$

غير موجودة

① $y' = \sec^2 x = 2$

② $y' = 3x \sin 3x + \cos 3x = -3\pi(0) - 1 = -1$

③ $2x + 2yy' = 0 \rightarrow y' = \frac{-x}{y} = \frac{-\sqrt{5}}{0}$ غير موجودة

Lecture 52

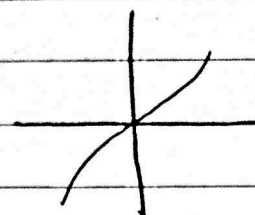
$$\frac{e^x + e^{-x}}{2} \quad \frac{e^x - e^{-x}}{2}$$

Hyperbolic Function

① Hyperbolic sine

سینہ الی

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$D_{\sinh} = \mathbb{R}$$

$$R_{\sinh} = \mathbb{R}$$

$$\sinh(0) = 0$$

$$\lim_{x \rightarrow +\infty} \sinh x = +\infty$$

$$\lim_{x \rightarrow -\infty} \sinh x = -\infty$$

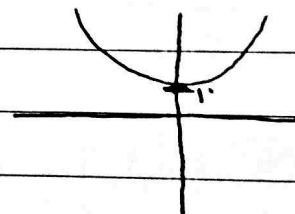
$$\frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

odd function

$$\sinh x \geq 0 \quad \forall x \geq 0, \quad \sinh x \leq 0 \quad \forall x \leq 0$$

② Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



$$D_{\cosh} = \mathbb{R}$$

$$R_{\cosh} = [1, \infty)$$

$$\lim_{x \rightarrow +\infty} \cosh x = +\infty$$

$$\lim_{x \rightarrow -\infty} \cosh x = +\infty$$

even function

$$\cosh(0) = 1$$

$$\frac{e^0 + e^{-0}}{2} = \frac{1+1}{2}$$

$$\cosh x \geq 1 \quad \forall x$$

③ Hyperbolic tangent

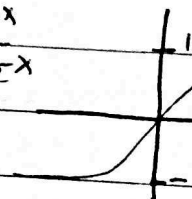
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$D_{\tanh} = \mathbb{R}$$

$$R_{\tanh} = (-1, 1)$$

$$\lim_{x \rightarrow +\infty} \tanh x = 1$$

$$\lim_{x \rightarrow -\infty} \tanh x = -1$$



$$x > 0$$

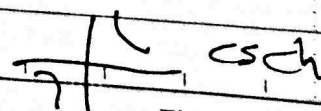
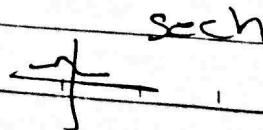
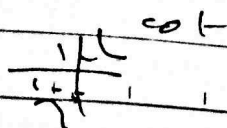
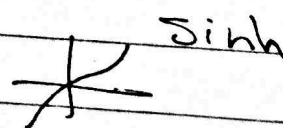
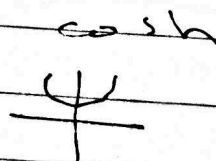
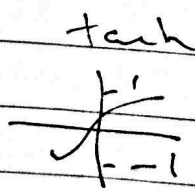
$$0 \leq \tanh x \leq 1$$

$$x < 0$$

$$-1 < \tanh x \leq 0$$

$$\tanh(0) = 0$$

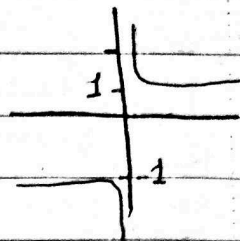
odd function



④ Hyperbolic cotangent $\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$D\coth x = \mathbb{R} \setminus \{0\}$ $R\coth x = (-\infty, -1) \cup (1, \infty)$

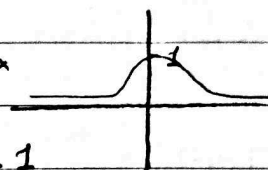
$\lim_{x \rightarrow +\infty} \coth x = 1$ $\lim_{x \rightarrow -\infty} \coth x = -1$



⑤ Hyperbolic secant $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

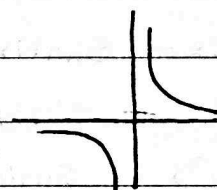
$D\operatorname{sech} x = \mathbb{R}$, $R\operatorname{sech} x = (0, 1]$

$\lim_{x \rightarrow +\infty} \operatorname{sech} x = 0$ $\lim_{x \rightarrow -\infty} \operatorname{sech} x = 0$ $0 < \operatorname{sech} x \leq 1$



⑥ Hyperbolic cosecant

$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$



$D\operatorname{csch} x = \mathbb{R} \setminus \{0\}$ $R\operatorname{csch} x = \mathbb{R} \setminus \{0\}$

$\lim_{x \rightarrow +\infty} \operatorname{csch} x = 0$ $\lim_{x \rightarrow -\infty} \operatorname{csch} x = 0$

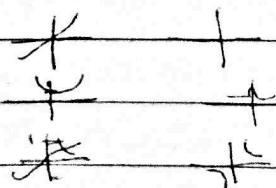
* Remark :-

① $\cosh x$, $\operatorname{sech} x$ even function

$\cosh(-x) = \cosh x$ $\operatorname{sech}(-x) = \operatorname{sech} x$

② $\sinh x$, $\tanh x$, $\coth x$, $\operatorname{csch} x$, odd function

$\left\{ \begin{array}{l} \sinh(-x) = -\sinh x, \tanh(-x) = -\tanh x \\ \operatorname{csch}(-x) = -\operatorname{csch} x, \coth(-x) = -\coth x \end{array} \right\}$



x	f	f'
	$\sinh x$	$\cosh x$
$\rightarrow$	$\cosh x$	$\sinh x$
	$\tanh x$	$\operatorname{sech}^2 x$
	$\coth x$	$-\operatorname{csch}^2 x$
	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
	$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

* Theorem :-

- ① $\cosh x + \sinh x = e^x$
- ② $\cosh x - \sinh x = e^{-x}$
- ③ $\cosh^2 x - \sinh^2 x = 1$
- ④ $1 - \tanh^2 x = \operatorname{sech}^2 x$ $\operatorname{sech}^2 x + \tanh^2 x = 1$
- ⑤ $\coth^2 x - 1 = \operatorname{csch}^2 x$ $\coth^2 x = \operatorname{csch}^2 x + 1$
- ⑥ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- ⑦ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- ⑧ $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
- ⑨ $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$ $\cosh$
- ⑩ $\cosh(2x) = \cosh^2 x + \sinh^2 x$
- ⑪ $\cosh(2x) = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$
- ⑫ $\sinh^2 x = \cosh(2x) - 1$
- ⑬ $\cosh^2 x = \frac{\cosh(2x) + 1}{2}$

$$\cos^2 x - \sin^2 x$$

$$2\cos^2 x - 1$$

$$1 - 2\sin^2 x$$

Lecture 53

Example:-

① Find $\sinh(2\ln 2) = \sinh(\ln 2^2) = \sinh(\ln 4)$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$$\frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - e^{\ln 1/4}}{2} = \frac{4 - \frac{1}{4}}{2} = \frac{16-1}{4 \times 2} = \frac{15}{8}$$

② Find $\cosh(\ln x) = \cosh(\ln x)$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$\frac{e^{\ln x} + e^{-\ln x}}{2} = \frac{e^{\ln x} + e^{\ln x^{-1}}}{2} = \frac{e^{\ln x} + e^{\ln \frac{1}{x}}}{2}$$

$$\frac{x + \frac{1}{x}}{2} = \frac{x^2 + 1}{2x}$$

③ If $\sinh x + \cosh x = 5$ Find x

$\sinh x + \cosh x = e^x$

$e^x = 5 \quad \ln e^x = \ln 5 \quad x = \ln 5$

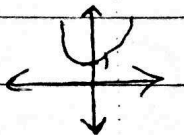
④ Given that $\operatorname{csch} x = \frac{1}{3}$ Find $\sinh x, \cosh x, \coth x, \tanh x, \operatorname{sech} x$

$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{-1}{3} \quad \sinh x = -3 \quad \cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \sinh^2 x = 1 \rightarrow \cosh^2 x - (-3)^2 = 1$

$\cosh^2 x - 9 = 1 \rightarrow \cosh^2 x = 10 \rightarrow \cosh x = \pm \sqrt{10} \neq$

$\therefore \cosh x = \sqrt{10} > 1$



$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-3}{\sqrt{10}} \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\sqrt{10}}$

$\coth x = \frac{\cosh x}{\sinh x} = \frac{-\sqrt{10}}{3}$

Example - find $\frac{dy}{dx}$ for

f	f'
$\sinh x$	$\cosh x$

① $y = \sinh(4x-8)$

$\sinh(g(x))$	$g'(x) \cosh(g(x))$
$\cosh(g(x))$	$g'(x) \sinh(g(x))$
$\tanh(g(x))$	$g'(x) \operatorname{sech}^2(g(x))$
$\operatorname{sech}(g(x))$	$-g'(x) \operatorname{sech}(g(x)) \tanh(g(x))$

$$\frac{dy}{dx} = 4 \cosh(4x-8)$$

② $y = \cosh^3(2x) = (\cosh(2x))^3$

$$\frac{dy}{dx} = 3(\cosh(2x))^2 \cdot 2 \cdot \sinh(2x)$$

$$= 6 \sinh(2x) \cosh^2(2x)$$

③ $y = \ln(\tanh(2x))$

$$\frac{dy}{dx} = \frac{2 \cdot \operatorname{sech}^2(2x)}{\tanh(2x)}$$

④ $y = \operatorname{sech}(e^x)$

$$\frac{dy}{dx} = -2e^x \operatorname{sech}(e^x) \tanh(e^x)$$

⑤ $y = \sinh(\cos^3 x) = \frac{dy}{dx} = -3 \sin(3x) \cdot \cosh(\cos(3x))$

⑥ $y = (\cosh x)^{\sinh x} = \ln y = \ln(\cosh x) \sinh x$

$$= \ln y = \sinh x \ln(\cosh x) = \frac{y'}{y} = \cosh x \cdot \ln(\cosh x) + \sinh x \frac{\sinh x}{\cosh x}$$

$$y' (\cosh x \ln(\cosh x) + \frac{\sinh^2 x}{\cosh x}) \rightarrow (\cosh x)^{\sinh x}$$

if $y(0) = \sinh 0 = 0$
 $\cosh 0 = 1$

$$(7) y = \frac{(\sinh x)(\ln x)(\tan^{-1} x)}{\cos(2x)}$$

$$\ln y = \ln \left(\frac{\sinh x \cdot \ln x \cdot \tan^{-1} x}{\cos 2x} \right)$$

$$\ln y = \ln(\sinh x) + \ln(\ln x) + \ln(\tan^{-1} x) - \ln(\cos 2x)$$

$$\frac{y'}{y} = \left(\frac{\cosh x}{\sinh x} + \frac{\frac{1}{x}}{\ln x} + \frac{\frac{1}{1+x^2}}{\tan^{-1} x} - \frac{-2 \sin(2x)}{\cos(2x)} \right)$$

$$y' = \coth x + \frac{1}{x \ln x} + \frac{1}{(1+x^2) \tan^{-1} x} + 2 \tan(2x) \left(\frac{\sinh x \cdot \ln x \cdot \tan^{-1} x}{\cos(2x)} \right)$$

lecture 54

Applications of Differentiation

$f(x)$ (Find domain $f(x)$)

$f'(x)$ f' also the slope of the tangent line f' also the rate of change of f
 $f''(x)$ f'' also the slope of the tangent line to f' f'' also the rate of change of f'
 a critical number ($a \in D_f$) $f'(a) = 0$ has a horizontal tangent line $f'(a) \neq 0$ has a non-horizontal tangent line
 $f'(a) \neq 0$ f' also the slope of the tangent line f' also the rate of change of f

f Increasing f' positive

If $x_2 > x_1 \rightarrow f(x_2) > f(x_1)$ $f' > 0$

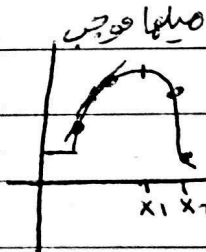
f Decreasing f' negative

If $x_2 > x_1 \rightarrow f(x_2) < f(x_1)$

$f' < 0$

f constant $f' = 0$

$f(x_2) = f(x_1) \forall x$



Concavity
 $f'' > 0$ f concave upward (the graph of f lies above all of its tangent)
 $f'' < 0$ f concave downward (the graph of f lies below all of its tangent)

Inflection Point -
 a point where the concavity changes

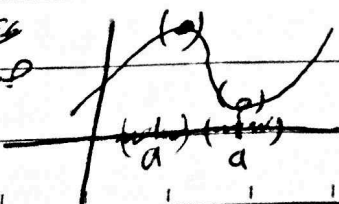
(an interval) $a \in D_f$
 1) f has a local max at $x = a$
 2) f has a local min at $x = a$
 3) the curve changes from concave up to concave down or vice versa

$f'' > 0$ f concave upward
 $f'' < 0$ f concave downward

$f'' > 0$ f concave upward
 $f'' < 0$ f concave downward

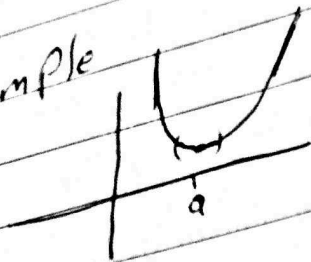
local max if $f(a) \geq f(x)$ when x is near a
 local min if $f(a) \leq f(x)$ when x is near a

absolute max if $f(a) \geq f(x) \forall x \in D_f$
 absolute min if $f(a) \leq f(x) \forall x \in D_f$



Example

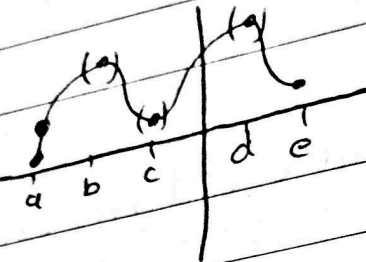
(1)



$D_f = \mathbb{R}$

at $x=a$ absolute min
No absolute max

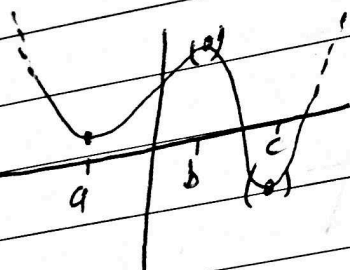
(2)



$D_f = [a, e]$

at $x=c$ local min = $f(c)$
at $x=b$ local max = $f(b)$
at $x=d$ local max = $f(d)$
at $x=d$ absolute max = $f(d)$
at $x=a$ absolute min = $f(a)$

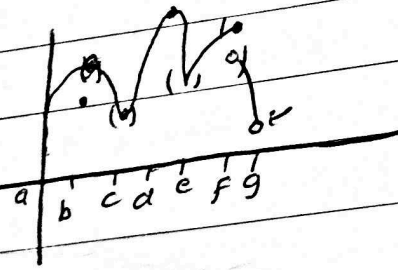
(3)



$D_f = \mathbb{R}$

at $x=a$ local min = $f(a)$
 $x=c$ local min = $f(c)$
at $x=b$ local max = $f(b)$
No absolute max

(4)



$D_f = [a, g]$

at $x=c$ absolute min = $f(c)$
($f(a)$ is not the absolute min)
at $x=b$ local min = $f(b)$
at $x=e$ local min = $f(e)$
at $x=b$ local max = $f(b)$
at $x=d$ local max = $f(d)$
at $x=f$ local max = $f(f)$
at $x=d$ absolute max = $f(d)$
No absolute min

* The extreme value theorem:-

If f is continuous on closed interval $[a, b]$, then f attains an absolute value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $(c \text{ and } d)$ in $[a, b]$

* Fermat's Theorem:-

If f has a local maximum or minimum at c and $f'(c)$ exists then $f'(c) = 0$

* Theorem:-

If f has a local maximum or minimum at c is a critical number of f

EXERCISE

Find $\frac{dy}{dx}$ for

$$y = (x^x)^{\frac{1}{2}} = y = x^{\frac{x}{2}} \rightarrow$$

$$\textcircled{a} f(x) = \sqrt{x^x} \quad y' = y \left(\frac{1}{2} \ln x + \frac{x}{2} \cdot \frac{1}{x} \right) = \sqrt{x^x} \left(\frac{1}{2} \ln x + \frac{1}{2} \right)$$

$$\rightarrow (\sin(\cos(\sqrt{\sinh \pi x})))^2$$

$$\textcircled{b} y = \sin^2(\cos(\sqrt{\sinh \pi x})) = -2 \cdot \sin(\cos(\sqrt{\sinh \pi x})) \cdot \cos(\cos(\sqrt{\sinh \pi x}))$$

$$\frac{\sin \sqrt{\sinh \pi x}}{\sin \sqrt{\sinh \pi x}} \cdot \frac{\pi \cosh(\pi x)}{2 \sqrt{\sinh \pi x}}$$

② Find a such that $y = a \sin(3t)$ satisfies $y'' + 2y = 4 \sin(3t)$

$$y' = 3a \cos(3t) \rightarrow y'' = -9a \sin(3t)$$

$$y'' + 2y = 4 \sin(3t) \rightarrow -9a \sin(3t) + 2a \sin(3t) = 4 \sin(3t)$$

$$-7a \sin(3t) = 4 \sin(3t) \rightarrow -7a = 4 \rightarrow a = -\frac{4}{7} \quad \#$$

③ If $g(x) = x f(x)$, $f(3) = 4$, $f'(3) = -2$ Find the equation of tangent line of $g(x)$ at $x = 3$

$$\text{point} = (3, g(3)) = (3, 3f(3)) = (3, 3(4)) = (3, 12)$$

$$\text{slope} = g'(x) = f(x) + x f'(x) \rightarrow g'(3) = f(3) + 3f'(3) \rightarrow 4 + (3)(-2) = -2$$

$$\text{equation} \rightarrow y - g(3) = g'(3)(x - 3) \rightarrow y - 12 = -2(x - 3)$$

$$\text{of tangent line} \quad y =$$

④ If $f(x) = \frac{1}{2-x}$ Find $f^{(n)}(x)$?

$$f'(x) = \frac{+1}{(2-x)^2} \rightarrow f''(x) = \frac{(2) \cdot 1 \cdot (2-x)(-1)}{(2-x)^4} = \frac{2}{(2-x)^3} = \frac{2!}{(2-x)^3}$$

$$f'''(x) = \frac{+2(3)(2-x)^2}{(2-x)^6} = \frac{6}{(2-x)^4} \quad f^{(4)}(x) = \frac{+6(4)(2-x)^3}{(2-x)^8}$$

$$\frac{24}{(2-x)^5} \quad f^{(5)}(x) = \frac{(24)(5)(2-x)^4}{(2-x)^{10}} = \frac{120}{(2-x)^6}$$

$$n! = n(n-1)(n-2) \dots$$

$$3! = 3(2)(1)$$

$$5! = (5)(4)(3)(2)(1)$$

$$f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$$

⑤ Show that $\frac{d}{dx} \left(\sqrt{\frac{1+\tanh x}{1-\tanh x}} \right) = \frac{1}{2} e^{x/2}$

$$y = \sqrt{\frac{1+\tanh x}{1-\tanh x}} = \left(\frac{1+\tanh x}{1-\tanh x} \right)^{1/4} = \left(\frac{1 + \frac{\sinh x}{\cosh x}}{1 - \frac{\sinh x}{\cosh x}} \right)^{1/4}$$

$$\left(\frac{\cosh x + \sinh x}{\cosh x - \sinh x} \right)^{1/4} = \left(\frac{\cosh x + \sinh x}{\cosh x} \cdot \frac{\cosh x}{\cosh x - \sinh x} \right)^{1/4}$$

$$\frac{\cosh x + \sinh x}{\cosh x - \sinh x} = y = \left(\frac{e^x}{e^{-x}} \right)^{1/4} = (e^{x+x})^{1/4} = e^{\frac{2x}{4}} = e^{x/2}$$

$$y' = \frac{1}{2} e^{x/2}$$

55 lecture

Example:-

① $f(x) = x^3 - 3x^2 + 5$ ② $f'(x) = 0 \rightarrow 3x^2 - 6x = 0 \rightarrow 3x(x-2)$

$Df = \mathbb{R}$

critical numbers $x=0$ $x=2$

Critical Points = $(0, 5)$ $(2, 1)$



f Increasing $(-\infty, 0) \cup (2, \infty)$

f decreasing $(0, 2)$

local, Relative

at $x=0$ local max: $f(0)=5$, $x=2$ local min: $f(2)=1$

No absolute

③ $f'(x) = 6x - 6 \rightarrow x=1$ Point(s) $(1, 3)$ $f'' = 6$

f concave upward $(1, \infty)$ f concave down $(-\infty, 1)$

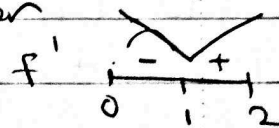
* ① $1 \in Df = \mathbb{R}$ ② f cont at $x=1$ ③ f'' changes about $x=1$

$\therefore (1, 3)$ inflection Point - نقطة انعطاف

④ $f(x) = x^4 - 4x$, $x \in [0, 2]$ $Df = [0, 2]$

$f'(x) = 4x^3 - 4 \rightarrow 4(x^3 - 1) \rightarrow x=1$ critical number

points $(1, -3)$



f Decreasing $(0, 1)$

f Increasing $(1, 2)$ at $x=1$ local min: $f(1) = -3$

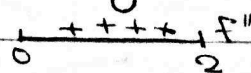
end points $(0, 0)$ $(2, 1)$ at $x=1$ absolute min: $f(1) = -3$

at $x=2$ absolute max: $f(2) = 1$ (local max) النظرية (✓) مثال دورتها الكبرى

$f''(x) = 12x^2 = 0$

f concave up $(0, 2)$

* No inflection points



③ $f(x) = xe^{-x}$ $Df = \mathbb{R}$

$f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x) = 0$

$e^x \neq 0$ $\boxed{x=1}$ $(1, e^{-1})$ critical Point

f' $\begin{array}{c} +++ \\ 1 \\ --- \end{array}$ f increasing $(-\infty, 1)$
 f decreasing $(1, \infty)$

at $x=1$ (local + absolute) $\max = e^{-1}$

$f''(x) = -e^{-x}(e^{-x} + x(-e^{-x})) = -2e^{-x} + xe^{-x} = 0$

$e^{-x}(-2+x) = 0$ $e^{-x} \neq 0$ $\boxed{x=2}$ $(2, 2e^{-2})$

f concave up $(2, \infty)$

f concave down $(-\infty, 2)$

f'' $\begin{array}{c} \cap \\ - \\ 2 \\ \cup \\ + \end{array}$

at $(2, 2e^{-2})$ inflection (① $2 \in Df$ ② f concave up at $x=2$ ③ $\cap \cup$)

Hw

④ $f(x) = \sqrt{9-x^2}$ $Df \in [-3, 3]$

Domain $9-x^2 \rightarrow x = \pm 3$ $\begin{array}{c} - + - \\ -3 \quad 3 \end{array}$

$f'(x) = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}} = 0$ $\boxed{x=0}$ $(0, -3) \cup (0, 3)$

(local + absolute) $\min$ at $\boxed{x=-3} = 0$

(local + absolute) $\max$ at $\boxed{x=3} = 0$

f increasing $(-3, 3)$ $-2x$

$f''(x) = \frac{(-1)(\sqrt{9-x^2}) + 9(\sqrt{9-x^2})^1}{(9-x^2)} = \frac{-\sqrt{9-x^2} - \frac{x}{\sqrt{9-x^2}}}{9-x^2}$

$\frac{-(9-x^2) - x}{\sqrt{9-x^2}(9-x^2)} = \frac{x^2 - x - 9}{\sqrt{9-x^2}(9-x^2)} = 0$ $x = \frac{1 \pm \sqrt{37}}{2}$

f concave up $(-3, \frac{1-\sqrt{37}}{2})$ f concave down $(\frac{1+\sqrt{37}}{2}, 3)$ -2.5 3.5
 at $(\frac{1-\sqrt{37}}{2}, 1.6)$ inflection

Lecture 56

Example: $f(x) = \sqrt{9-x^2}$ Find the interval of concavity

$$Df = [-3, 3] \quad f'(x) = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}}$$

$$f''(x) = \frac{-\sqrt{9-x^2}}{(\sqrt{9-x^2})^2} - \frac{-2x}{2\sqrt{9-x^2}} (-x) = \frac{-\sqrt{9-x^2}}{1} - \frac{x^2}{\sqrt{9-x^2}}$$

$$= \frac{-9-x^2+x^2}{\sqrt{9-x^2}} = \frac{-9}{(9-x^2)^{3/2}} = \frac{-9}{(9-x^2)^3}$$

$$= \frac{-9-x^2+x^2}{\sqrt{9-x^2}} \div (9-x^2) = \frac{-9}{(9-x^2)^{3/2}} = \frac{-9}{(9-x^2)^3}$$

$$(9-x^2)^{3/2} = 0 \quad x = \pm 3$$

at $x = -3, 3$ critical numbers

$$-3 \quad 0 \quad 3$$

domain is

concave down

$$f'' \quad -3 \quad 3$$

concave down = $(-3, 3)$

Example: $f(x) = \frac{1}{x^3}$ Find the intervals of increasing and decreasing.

$$Df = \mathbb{R} \setminus \{0\}$$

$$f(x) = x^{-3} \rightarrow f'(x) = -3x^{-4} = \frac{-3}{x^4} \rightarrow x^4 = 0 \notin Df$$

f' decreasing $\mathbb{R} \setminus \{0\}$

Example: $f(x) = x^2 - x - \ln x$ Find the critical numbers of function

$$Df = (0, \infty) \quad f'(x) = 2x - 1 - \frac{1}{x} = 2x(\frac{x}{x}) - 1(\frac{x}{x}) - \frac{1}{x}$$

$$f'(x) = \frac{2x^2 - x - 1}{x} \quad 2x^2 - x - 1 = 0 \quad x = \frac{1 \pm \sqrt{1+4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{-7}}{4} \rightarrow \frac{1 \pm i\sqrt{7}}{4}$$

$$x=0 \rightarrow x=0 \notin Df$$

$(1, 0)$ critical point

Example:- $f(x) = (x+1)^5(x-2)^4$ Find the critical numbers of f

$$Df = \mathbb{R} \quad f'(x) = 5(x+1)^4(x-2)^4 + (x+1)^5(4(x-2)^3) = 0$$

$$(x+1)^4(x-2)^3(5(x-2) + 4(x+1)) = 0$$

$$(x+1)^4(x-2)^3(5x-10+4x+4) = 0$$

$$(x+1)^4(x-2)^3(9x-6) = 0$$

$$(x+1)^4 = 0 \rightarrow x = -1 \quad (x-2)^3 = 0 \rightarrow x = 2$$

$$9x-6=0 \rightarrow x = \frac{6}{9} = \frac{2}{3}$$

critical numbers

critical points $(-1, f(-1))$, $(2, f(2))$, $(\frac{2}{3}, f(\frac{2}{3}))$

lecture 57

Example :- $f(x) = \cos x + \sin x$, $x \in [0, 2\pi]$

Find the intervals of increasing and decreasing and local absolute max and min

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

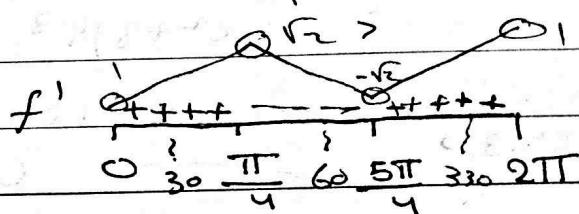
$$f'(x) = -\sin x + \cos x = 0 \rightarrow \sin x = \cos x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \quad \left(\frac{\pi}{4}, \sqrt{2} \right), \left(\frac{5\pi}{4}, -\sqrt{2} \right)$$

critical points

$$(0, 1) \quad (2\pi, 1)$$

end points



at $x = \frac{\pi}{4}$ local max $= f(\frac{\pi}{4}) = \sqrt{2}$ ★

$x = \frac{5\pi}{4}$ local min $= f(\frac{5\pi}{4}) = -\sqrt{2}$

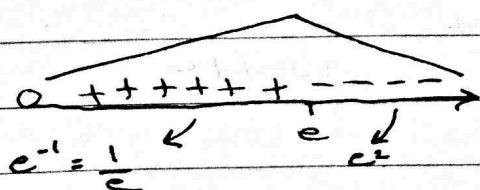
at $x = \frac{\pi}{4}$ absolute max $= \sqrt{2}$

$x = \frac{5\pi}{4}$ absolute min $= -\sqrt{2}$

Example :- $f(x) = \ln x$ Find the intervals of increasing and decreasing.

$$Df = (0, \infty)$$

$$f'(x) = \frac{1}{x} \cdot x - \ln x = \frac{1 - \ln x}{x^2} \rightarrow \begin{matrix} \ln x = 1 \rightarrow x = e \in Df \\ \ln x = 0 \rightarrow x = 1 \in Df \end{matrix}$$



f increasing $(0, e)$

f decreasing (e, ∞)

Example 1: $f(x) = \frac{x^2}{x^2+3}$ Find the interval of concavity and inflection points

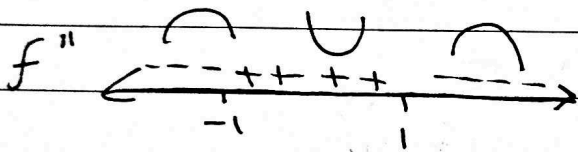
$$D_f = \mathbb{R} \quad f'(x) = \frac{2x(x^2+3) - 2x(x^2)}{(x^2+3)^2} = \frac{2x^3+6x-2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$$f''(x) = \frac{6(x^2+3)^2 + 6x \cdot 2 \cdot (x^2+3) \cdot (2x)}{(x^2+3)^4}$$

$$= \frac{(x^2+3)(6(x^2+3) + 24x^2)}{(x^2+3)^4} = \frac{6x^2+18-24x^2}{(x^2+3)^3}$$

$$\frac{18-18x^2}{(x^2+3)^3} \xrightarrow{\quad} 18(1-x^2) = 0 \rightarrow x = \pm 1$$

$$\xrightarrow{\quad} (x^2+3)^3 \neq 0$$



f concave up $(-1, 1)$

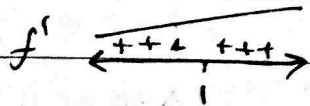
f concave down $(1, \infty) \cup (-\infty, -1)$

at $(-1, f(-1))$ inflection $(\textcircled{1} \pm 1 \in D_f \textcircled{2} f \text{ cont at } x = \pm 1)$
 $(1, f(1))$ $\textcircled{3} \frac{n}{-1} \frac{u}{1} \frac{n}{1}$

Example $f(x) = (x-1)^3$ find the interval increasing and decreasing

$$D_f = \mathbb{R}$$

$$f'(x) = 3(x-1)^2 = 0 \rightarrow x = 1$$



f increasing on $\mathbb{R}$

$(1, 0)$ critical point

local (no) absolute

Example: (ex) $|2x-8|$ find the critical number

$$Df = \mathbb{R} \quad f(x) = \begin{cases} 2x-8, & x > 4 \\ 8-2x, & x < 4 \end{cases} \rightarrow$$

$$f'(x) = \begin{cases} 2, & x > 4 \\ -2, & x < 4 \end{cases}$$

d.n.e $x=4$

$$\begin{array}{c} -2 \quad 2 \\ \leftarrow \quad \rightarrow \\ 4 \end{array}$$

$$f'_{+}(4) = 2 \quad f'_{-}(4) = -2 \quad f'(4) = \text{d.n.e}$$

at $x=4$ critical number

(4,0) critical point

##

Example: If at $x=1$ we have extrema values for $f(x) = ax^3 - 2x + 3$ find a ?

$$f'(1) = 0 \quad f'(x) = 3ax^2 - 2 \rightarrow 3a - 2 = 0 \rightarrow a = \frac{2}{3}$$

##

Example: If $(1,3)$ extrema point for

$$f(x) = x^2 + ax + b \quad \text{find } (a, b) ?$$

$$f(1) = 3 \quad f'(1) = 0$$

$$f'(x) = 2x + a$$

$$f'(1) = 2 + a = 0 \quad a = -2$$

$$f(1) = 1^2 - 2(1) + b = 3$$

$$\boxed{b = 4}$$

[H.W]

① If at $x=1$ inflection point for $f(x)=2ax^3-5x^2+1$
Find a ? $f''(1)=0$

$$f'(x)=6ax^2-10x \rightarrow f''(x)=12ax-10 \rightarrow 12a-10=0$$

$$a = \frac{10}{12}$$

② If $f(x)=ax^3-bx^2-qx+8$ has local max at $x=2$
and local min at $x=3$ Find q, b ?

$$f'(2)=0 \quad f'(3)=0$$

$$f'(x)=3ax^2-2bx-q \rightarrow 12a-4b-q=0$$

$$27a-6b-q=0$$

$$2-2b=0$$

$$2=2b \rightarrow b=1$$

$$-4\left(\frac{15}{2}\right)a-q=0 \quad 12a-30a-q=0 \rightarrow -18a-q=0$$

$$-9 \quad a=-0.5 \quad b=-3.75$$

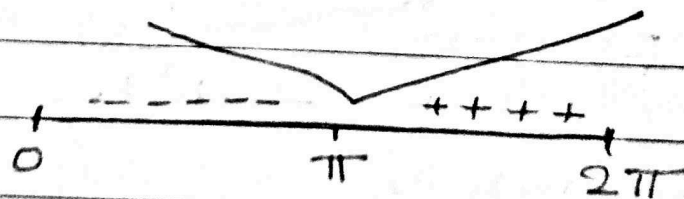
the intervals of increasing and decreasing
 $f(x)=1+2\cos x, x \in [0, 2\pi]$

$$\sin x = 0$$

$$x = 0, \pi$$

$$(0, \pi)$$

$$(\pi, 2\pi)$$



$$\frac{4}{5}, \frac{7}{5}$$

$$\frac{10}{15} = \frac{2}{3}$$

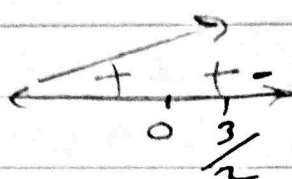
$$\frac{3}{5} \times x = \frac{-\frac{2}{5}}{\frac{1}{5}}$$

④ Find the intervals of increasing and decreasing
 $f(x) = 4x^{5/3} - x^{8/3}$ DS: IR

$$\frac{4}{5}x^{2/3} - \frac{8}{3}x^{5/3}$$

$$f'(x) = 4\left(\frac{3}{5}\right)x^{-2/3} - \frac{8}{3}x^{5/3} = \frac{12}{5}x^{-2/3} - \frac{8}{3}x^{5/3}$$

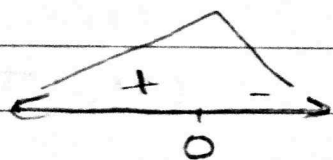
$$x=0$$



f increasing
 f decreasing

⑤ find the intervals of increasing and decreasing
 + intervals of concavity + local and absolute max and
 and inflection point and critical point for $f(x) = x^{2/3} - x$

$$f'(x) = \frac{2}{3}x^{-1/3} - 1 = 0 \quad x\left(\frac{2}{3}x^{-4/3} - 1\right) \quad x=0,$$

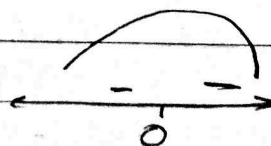


f increasing $(-\infty, 0)$
 f decreasing $(0, \infty)$

local max, absolute max at $x=0 = 0$
 No SW

point $(0, 0)$ cr
 Po

$$f''(x) = \frac{-2}{9}x^{-4/3} - 1$$



No inflection



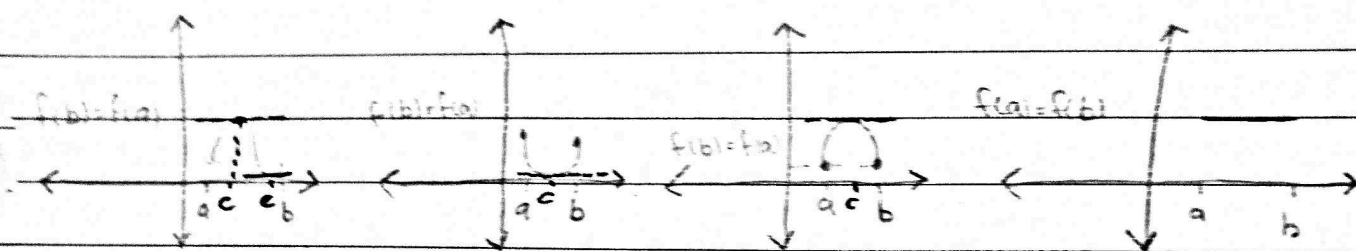
Lecture 59

* Rolle's Theorem: - let $y=f(x)$ that satisfies

- ① f continuous on $[a, b]$
- ② f differentiable on (a, b)
- ③ $f(a) = f(b)$

Then $\exists c \in (a, b)$ such that $f'(c) = 0$

there exist



(1)

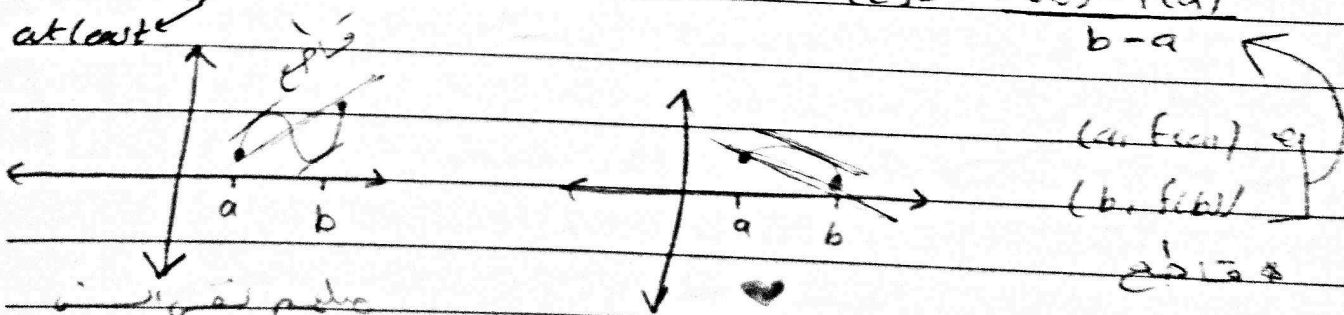
at least $\exists c$ (one c)

The Mean Value Theorem: - let $y=f(x)$ that satisfies (M.V.T)

- ① f continuous on $[a, b]$
- ② f differentiable on (a, b)

Then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

at least



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example - Find the values of c that satisfy the inclusion of M.V.T

① $f(x) = \frac{1}{x-1}$ on $[2, 5]$

① f cont on $[2, 5]$

② $f'(x) = \frac{-1}{(x-1)^2}$ diff on $(2, 5)$

$\frac{1}{(c-1)^2} = \frac{1}{(5-1)^2} = \frac{1}{16}$
 $\frac{1}{(c-1)^2} = \frac{1}{(2-1)^2} = 1$
 $\frac{1}{(c-1)^2} = \frac{1}{(5-1)^2} = \frac{1}{16}$
 $\frac{1}{(c-1)^2} = \frac{1}{(2-1)^2} = 1$
 $(c-1)^2 = 4$
 $c-1 = 2 \Rightarrow c = 3$
 $c-1 = -2 \Rightarrow c = -1$

1 is diff

then by M.V.T $\exists$ at least $c \in (2, 5)$ s.t

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{\frac{1}{4} - 1}{3} = \frac{-\frac{3}{4}}{3} = -\frac{1}{4}$$

$$(c-1)^2 = 4 \quad |c-1| = 2 \rightarrow c-1 = 2 \quad c-1 = -2$$

$c = 3$ #

$c = -1$ #

موجوب بالفترة ✓

غير موجود بالفترة ✗

② $f(x) = \sqrt{25-x^2}$ on $[-3, 5]$

① f cont $[-3, 5]$ --

② $f'(x) = \frac{-2x}{2\sqrt{25-x^2}} = \frac{-x}{\sqrt{25-x^2}}$ diff on $(-3, 5)$

then $\exists$ at least $c \in (-3, 5)$ such that

$$f'(c) = \frac{f(5) - f(-3)}{5 - (-3)} = \frac{0 - 4}{8} = -\frac{1}{2}$$

$$2c = \sqrt{25-c^2}, \quad 4c^2 = 25-c^2 \rightarrow 5c^2 = 25 \rightarrow c^2 = 5 \rightarrow c = \pm\sqrt{5}$$

$c = \sqrt{5}$ ✓ $c = -\sqrt{5}$ ✗

$-2\sqrt{5} = \sqrt{25-5}$ ✗

lecture 60

لوسبال

Indeterminate forms and L'Hospital's Rule :

كمان عند صفر

$$\textcircled{1} \frac{0}{0}, \frac{+\infty}{+\infty}, \frac{+\infty}{-\infty}, \frac{-\infty}{+\infty}, \frac{-\infty}{-\infty} :$$

Examples

$$\textcircled{1} \lim_{x \rightarrow +\infty} \frac{e^{3x}}{x^2} \quad \lim_{x \rightarrow +\infty} e^{3x} = +\infty, \quad \lim_{x \rightarrow +\infty} x^2 = +\infty$$

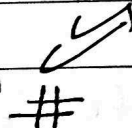
$$\frac{+\infty}{+\infty}$$

نشتت البسط

$$\lim_{x \rightarrow +\infty} \frac{3e^{3x}}{2x} \quad \lim_{x \rightarrow +\infty} 3e^{3x} = +\infty, \quad \lim_{x \rightarrow +\infty} x = +\infty$$

نشتت المقام

$$\lim_{x \rightarrow +\infty} \frac{9e^{3x}}{2} = \frac{9}{2} \lim_{x \rightarrow +\infty} e^{3x} = +\infty$$



$$\textcircled{2} \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$$

$$\lim_{x \rightarrow 0^+} \ln(\sin x) = \ln(\lim_{x \rightarrow 0^+} \sin x) = -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{\cos x}{\sin x}$$

$$\lim_{x \rightarrow 0^+} \ln(\tan x) = \ln(\lim_{x \rightarrow 0^+} \tan x) = -\infty$$

$$\frac{\sec^2 x}{\tan x}$$

$$\lim_{x \rightarrow 0^+} \frac{\frac{1}{\tan x}}{\frac{\sec^2 x}{\tan x}}$$

$$\lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x} = \lim_{x \rightarrow 0^+} \frac{1}{\frac{1}{\cos^2 x}} = \lim_{x \rightarrow 0^+} \cos^2 x = 1$$

(*)

② $0 \cdot \pm \infty$, $+\infty - \infty$, $-\infty + \infty$:

$+\infty + \infty = +\infty$
 $-\infty - \infty = -\infty$
 $\neq$

Example:

$0 \cdot -\infty$

① $\lim_{x \rightarrow 0^+} x \ln x$

$\frac{-\infty}{+\infty} \#$

$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$

نخرج للحالة ①

$\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$

$\lim_{x \rightarrow 0^+} x = 0$

$\lim_{x \rightarrow 0^+} \ln x = -\infty$

$\lim_{x \rightarrow 0^+} x = -\infty$

$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$

$\frac{a}{b} = a \cdot \frac{1}{b}$
 $a \cdot b = \frac{a}{\frac{1}{b}} = \frac{1}{\frac{1}{a \cdot b}}$
 $a \cdot b = \frac{1}{\frac{1}{a} \cdot \frac{1}{b}}$
 $x = e^{\ln x}$
 $x = \ln e^x$
 $\frac{a}{b} + \frac{c}{d} = \frac{ad+cd}{bd}$
 $\sec x = \frac{1}{\cos x}$

$\lim_{x \rightarrow 0^+} \frac{1}{x} \div \frac{-1}{x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot x^2$

$\lim_{x \rightarrow 0^+} -x = 0$

$+\infty - \infty$

② $\lim_{x \rightarrow \frac{\pi}{2}^-} (\sec x - \tan x)$

نغير في شكل السؤال

$\lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1 - \sin x}{\cos x} \right) \frac{0}{0}$

رجعنا ①

$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec x = \frac{1}{\cos x}$
 $\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = \frac{\sin x}{\cos x}$
 $\frac{++}{\frac{\pi}{2}}$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x$

CS (0.1)

$\frac{1}{\frac{\pi}{2}}$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} = \frac{0}{1} = 0 \neq$

نقطة النظر (قيمة غير محددة)

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec(3x) = \frac{1}{\cos(3x)} \rightarrow +\infty$$

$\infty \cdot 0$

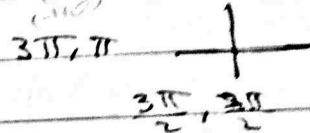
$$(3) \lim_{x \rightarrow \frac{\pi}{2}^-} \sec(3x) \cos(5x)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos(5x)}{\cos(3x)}$$

$\frac{0}{0}$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \cos(5x) = 0$$

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$



$$\frac{-5 \sin(5x)}{-3 \sin(3x)} = \frac{5}{3} \frac{\sin(5x)}{\sin(3x)}$$

$$= \frac{5}{3} \frac{(1)}{(-1)} = -\frac{5}{3}$$

$$(4) \lim_{x \rightarrow +\infty} (x - \ln(x^2 + 1))$$

$$\lim_{x \rightarrow +\infty} x = +\infty$$

$$\lim_{x \rightarrow +\infty} (\ln e^x - \ln(x^2 + 1))$$

$$\lim_{x \rightarrow +\infty} \ln(x^2 + 1) = +\infty$$

$$\lim_{x \rightarrow +\infty} \ln\left(\frac{e^x}{x^2 + 1}\right)$$

$$\ln\left(\lim_{x \rightarrow +\infty} \frac{e^x}{x^2 + 1}\right) \frac{+\infty}{+\infty}$$

$$\lim_{x \rightarrow +\infty} e^x = +\infty$$

$$\ln\left(\lim_{x \rightarrow +\infty} \frac{e^x}{x^2 + 1}\right) = \ln\left(\lim_{x \rightarrow +\infty} \frac{e^x}{x^2}\right) = +\infty$$

$$\lim_{x \rightarrow +\infty} x^2 + 1 = +\infty$$

$$(5) \lim_{x \rightarrow +\infty} (5^x - 2^x)$$

$$\lim_{x \rightarrow +\infty} b^x = +\infty$$

$$+\infty(1-0) = +\infty$$

$$b > 1$$

$$\lim_{x \rightarrow +\infty} 5^x \left(1 - \frac{2^x}{5^x}\right) \rightarrow \lim_{x \rightarrow +\infty} 5^x \left(1 - \left(\frac{2}{5}\right)^x\right)$$

$$\lim_{x \rightarrow +\infty} 5^x = +\infty$$

$$= +\infty$$

$$\lim_{x \rightarrow +\infty} \left(\frac{2}{5}\right)^x = 0$$

$$0 < b < 1$$

lecture 65

(3) $1^{+0}, 0^0, 0^0$

① $y = \lim$ قاطع

۱۴۳۰

٣) خيال ١٤٠٠ للمقدار الناتج

② الجواب = ③ الجواب هي

Example 1-2

$$\lim (1+3x) = 1$$

$$\lim_{x \rightarrow 0^+} (1 + 3x)^{-\frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \frac{-5}{x} = -\infty$$

$$y = (1 + 3x)^{-\frac{5}{x}}$$

$$\ln y = \ln(1+3x)^{\frac{-5}{x}}$$

$$\ln y = \frac{-5}{x} \ln(1+3x)$$

الكالم 2

- 20 - 0

$$\star \lim_{x \rightarrow 0^+} \frac{-5}{x} \cdot \ln(1+3x)$$

$$\lim_{x \rightarrow 0^+} \frac{-5}{x} = -\infty$$

$$\lim_{x \rightarrow 0^+} \ln(43x) = 0$$

$$\lim_{x \rightarrow 0^+} x = 0$$

$$\lim_{x \rightarrow 0^+} \frac{-5 \ln(1+3x)}{x}$$

$$\lim_{x \rightarrow 0^+} \frac{-5 \left(\frac{3}{1+3x} \right)}{1} = \lim_{x \rightarrow 0^+} \frac{-15}{1+3x} = -15$$

$$\lim_{x \rightarrow 0^+} (1+3x)^{-\frac{5}{x}} = \underline{\underline{e^{-15}}} \neq$$

② $\lim_{x \rightarrow 0^+} x^x$

$$y = x^x$$

2.131 $(0, -\infty)$

$$\lim_{x \rightarrow 0^+} 0^0$$

$$\ln y = \underline{x \ln x}$$

$$\lim_{x \rightarrow 0^+} x \cdot \ln x$$

① $\lim_{x \rightarrow \infty} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{+\infty}$

$$\rightarrow \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \frac{x^2}{-x} = \overset{\text{not}}{-x} = 0$$

$$\lim_{x \rightarrow 0^+} x^x = e^0 = 1$$

$$\frac{1}{\sqrt{x^2 + a^2}} = \ln x + c$$

Remark

$$\textcircled{1} \lim_{x \rightarrow +\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$$

$$\lim_{x \rightarrow 0} \left(1 + \frac{a}{x}\right) = 1$$

$$\lim_{x \rightarrow 0} bx = +\infty$$

$$\textcircled{2} \lim_{x \rightarrow 0} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$$

$$y = \left(1 + \frac{a}{x}\right)^{bx}$$

$$\ln y = \ln \left(1 + \frac{a}{x}\right)^{bx}$$

$$\ln y = bx \ln \left(1 + \frac{a}{x}\right) \quad \lim_{x \rightarrow 0} bx \ln \left(1 + \frac{a}{x}\right)$$

$$\frac{\lim_{x \rightarrow 0} \ln \left(1 + \frac{a}{x}\right)}{\frac{1}{bx}} = \frac{0}{0} = \frac{-\frac{a}{x^2}}{\frac{-b}{bx^2}} = \frac{-abx^2}{-bx^2 \ln \left(1 + \frac{a}{x}\right)}$$

$$\lim_{x \rightarrow 0} \frac{ab}{1 + \frac{a}{x}} = e^{ab} \neq$$

Example $\textcircled{1} \lim_{x \rightarrow +\infty} \left(1 + \frac{5}{x}\right)^{1x} = e^{5 \cdot 1} = e^5$

$$\textcircled{2} \lim_{x \rightarrow 0} (1 - x)^{\frac{3}{x}} = \lim_{x \rightarrow 0} (1 + -x)^{\frac{3}{x}} = e^{-3} = \frac{1}{e^3}$$

$$\textcircled{3} \lim_{x \rightarrow \infty} \left(\frac{x+4}{x}\right)^{5x} = \lim_{x \rightarrow \infty} \left(\frac{x}{x} + \frac{4}{x}\right)^{5x} = e^{20}$$

$$\textcircled{4} \lim_{x \rightarrow \infty} \left(\frac{x+2}{x+3}\right)^x = \lim_{x \rightarrow \infty} \frac{\left(\frac{x+2}{x}\right)^x}{\left(\frac{x+3}{x}\right)^x} = \frac{e^2}{e^3} = \frac{1}{e} \neq$$

lecture 62

Indefinite Integrals: $\int$ التكامل غير المحدود

$$\int f'(x) \cdot dx = \underbrace{f(x) + c}_{\text{antiderivative}}, \quad c \text{ constant} \quad \int \text{من التكامل}$$

Example:- Find general antiderivative of

① $h(x) = \sec^2 x$ Ans: $\tan x + c$ general antiderive $\int \sec^2 x \cdot dx = \tan x + c$

$$g(x) = \int \cos x + \frac{1}{1+x^2} \cos x + \sin x + \tan^{-1} x + c$$

Definite Integrals: التكامل المحدود

$$\int_a^b f'(x) dx = f(x) \Big|_a^b = f(b) - f(a)$$

① $\int k \cdot dx = kx + c$, k, c constant

Ex: $\int 5 dx = 5x + c$

② $\int x^r dx = \frac{x^{r+1}}{r+1} + c$, $r \neq -1$

Example: $\int_1^2 \frac{1}{\sqrt{x}} dx = \int_1^2 \frac{1}{x^{1/2}} dx = \int_1^2 x^{-1/2} dx$

$$\frac{x^{-1/2+1}}{-1/2} \Big|_1^2 = 2x^{1/2} \Big|_1^2 = 2\sqrt{2} - 2 \neq$$

$$\textcircled{2} \int \frac{5x^3 - x}{x} dx$$

$$\int (5x^2 - 1) dx = \frac{5x^3}{3} - x + c$$

$$\textcircled{3} \int \sqrt{x} (1+x) \cdot dx \quad \int \sqrt{x} + \sqrt{x^3} dx$$

$$\frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5} + c$$

$$\boxed{3} \int (ax+b)^r dx = \frac{(ax+b)^{r+1}}{a(r+1)} + c, r \neq -1$$

Example

$$\textcircled{1} \int (5x+3)^3 \cdot dx = \frac{(5x+3)^4}{5(4)} + c$$

$$\textcircled{2} \int_0^1 \frac{1}{\sqrt{7-3x}} \cdot dx = \int_0^1 (7-3x)^{-1/2} dx = \frac{(7-3x)^{1/2}}{-3(1/2)} + c$$

$$\frac{2(7-3x)^{1/2}}{-3} + c$$

lecture 6.3

$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$$

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$$

$$\int \sec(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$$

$$\int \csc(ax+b) dx = -\frac{1}{a} \cot(ax+b) + c$$

$$\int \sec(ax+b) \tan(ax+b) dx = \frac{1}{a} \sec(ax+b) + c$$

$$\int \csc(ax+b) \cot(ax+b) dx = -\frac{1}{a} \csc(ax+b) + c$$

Example: find

$$\textcircled{1} \int \sin(5x+7) dx = \frac{1}{5} \cos(5x+7) + c$$

$$\textcircled{2} \int 2 \sin x \cos x dx = \int \sin 2x dx \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$= \frac{1}{2} \cos 2x + c$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\textcircled{3} \int \frac{\sin(2x)}{\sin x} dx = \int \frac{\sin x \cos x \cdot 2}{\sin x} dx \quad \sin(2x) = 2 \sin x \cos x$$

$$2 \int \cos x dx = 2 \sin x + c$$

$$\boxed{5} \int c \cdot f(x) dx = c \int f(x) dx$$

$$\int (f+g) dx = \int f(x) dx + \int g(x) dx$$

$$\text{Ex } \int (\sec^2 x - 5 \cos(9x)) dx$$

$$\tan x - \frac{5}{9} \sin(9x) + c$$

$$\int \sin^2(3x) dx = \int \frac{1 - \cos(6x)}{2} dx$$

$$= \frac{1}{2} \int 1 - \cos 6x dx = \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + C$$

$$= \frac{x}{2} + \frac{1}{12} \sin 6x + C \quad \neq$$

$$\textcircled{3} \int \frac{\sec x + \cos x}{2 \cos x} dx = \frac{1}{2} \int \frac{\sec x}{\cos x} + \frac{\cos x}{\cos x} dx$$

$$= \frac{1}{2} \int \sec^2 x + 1 dx = \frac{1}{2} (\tan x + x) + C$$

$$\textcircled{4} \int \frac{2}{1 - \cos x} dx = 2 \int \frac{1}{1 - \cos x} \left(\frac{1 + \cos x}{1 + \cos x} \right) dx$$

$$\text{csc}^2 x + \cot x \csc x$$

$$2 \int \frac{1 + \cos x}{1 - \cos x} = 2 \int \frac{1 + \cos x}{\sin x} dx$$

$$\sin^2 x + \cos^2 x = 1$$

$$2 (-\cot x - \csc x) + C$$

$$\textcircled{5} \int \frac{\cos x}{\sin^2 x} dx = \int \cot x \csc x dx = -\csc x + C$$

$$\text{H.W} \int \cos^2 x dx = \int \frac{1 + \cos(2x)}{2}$$

$$= \frac{1}{2} \int 1 + \cos(2x) dx = \frac{1}{2} \left(x + \frac{1}{2} \sin(2x) \right) + C$$

lecture 64

$$\textcircled{6} \int_a^a f(x) \cdot dx = 0$$

$$\star \int_a^b c \, dx = c(b-a)$$

$$cx \Big|_a^b \quad cb - ca = c(b-a)$$

$$\star \int_a^b f(x) \, dx = - \int_b^a f(x) \cdot dx$$

$$\star \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \cdot dx \quad \overbrace{a \quad c \quad b}$$

$$\star \int_{-a}^a \text{odd function} = 0 \quad \#$$

$$\star \int_{-a}^a \text{even function} = 2 \int_0^a \text{even function}$$

Example: $\textcircled{1} \# \int_3^2 f(x) \, dx = 6$, $\int_3^2 (5-f(x)) \, dx = 4$

Find $\int_1^0 f(x) \, dx$?

$$\int_1^2 3f(x) \, dx = 6 \quad = \quad \frac{3}{3} \int_1^2 f(x) \, dx = \frac{6}{3} \rightarrow \boxed{\int_1^2 f(x) \, dx = 2} \quad \#$$

$$\int_3^2 (5-f(x)) \, dx = 4 \rightarrow \int_3^2 5 \, dx - \int_3^2 f(x) \, dx = 4 = 5(2-3) - \int_3^2 f(x) \, dx = 4$$

$$\boxed{+ \int_3^2 f(x) \, dx = -9} \quad \# \quad \int_1^2 f(x) \, dx + \int_2^3 f(x) \, dx = 2 + 9 = \boxed{11} \quad \#$$

$$\int_2^3 f(x) = -9 \quad - \int_2^3 f(x) = -9 \quad = \int_3^2 f(x) = 9$$

② Find $\int_{-3}^3 \frac{\sin(2x)}{x^4 + x^2 + 1} dx$

$f(-x) = -f(x)$ odd

$\frac{\sin(-2x)}{(-x)^4 + (-x)^2 + 1} = \frac{-\sin(2x)}{x^4 + x^2 + 1} = -f(x)$ odd function

$= 0$

③ If $\int_{-1}^a x dx = 0$ find a

$\frac{x^2}{2} \Big|_{-1}^a = 0 \quad \frac{a^2}{2} - \frac{1}{2} = 0 \quad \frac{a^2}{2} = \frac{1}{2} \quad a^2 = 1$

$a = \pm 1$

④ If $f(x) = \begin{cases} 2x, & 0 \leq x \leq 2 \\ x^2, & 2 < x \leq 3 \end{cases}$ find $\int_0^3 f(x) dx$

$\begin{array}{c} 2x \quad x^2 \\ 0 \quad 2 \quad 3 \end{array}$

$\int_0^3 f(x) dx = \int_0^2 2x dx + \int_2^3 x^2 dx$

$\frac{2x^2}{2} \Big|_0^2 + \frac{x^3}{3} \Big|_2^3 = (4-0) + (9-\frac{8}{3})$

H.W

$$(5) \int \frac{1}{x^5} dx \quad \int x^{-5} dx = \frac{-x^{-4}}{4} + c$$

$$(6) \int \sqrt{4-5x} dx \quad \int (4-5x)^{1/2} dx = \frac{-2(4-5x)^{3/2}}{5 \cdot 3} = \frac{-2(4-5x)^{3/2}}{15} + c$$

$$(7) \int (1 + \tan^2 x) dx \quad \int \sec^2 x dx = \tan x + c$$

$$\begin{cases} \sec^2 x = 1 + \tan^2 x \\ \csc^2 x = 1 + \cot^2 x \end{cases}$$

$$(8) \int_5^{10} 5 dx \quad 5(10-5) = 25$$

$$(9) \int_1^4 \frac{y^{1/2}}{y^2} dy \quad \int \frac{y^{1/2}}{y^2} = \frac{y}{y^2} dy \quad \int y^{-3/2} - y^{-1} dy$$

$$-2y^{1/2} - \ln y$$

$$(10) \int_0^{\pi/3} \frac{\sin \theta + \sin \theta \tan \theta}{\sec^2 \theta} d\theta$$

$$\int_0^{\pi/3} \frac{\sin \theta (1 + \tan^2 \theta)}{\sec^2 \theta} d\theta = \int_0^{\pi/3} \sin \theta d\theta = -\cos \theta \Big|_0^{\pi/3}$$

$$-(\frac{1}{2} - 1) = \frac{1}{2} \#$$

$$(11) \int_0^2 |x-1| dx \quad \int_0^1 1-x dx + \int_1^2 x-1 dx$$

$$\begin{matrix} 1-x & x-1 \\ 0 & 1 & 2 \end{matrix} \quad \left(x - \frac{x^2}{2} \right) \Big|_0^1 + \left(\frac{x^2}{2} - x \right) \Big|_1^2 =$$

$$\left(1 - \frac{1}{2} \right) + (2-2) - \left(\frac{1}{2} - 1 \right) = \frac{1}{2} + \frac{1}{2} = 1$$

lecture 6.5

$$\boxed{7} \int e^{ax+b} \cdot dx = \frac{e^{ax+b}}{a} + c$$

$$\int b^{ax+b} \cdot dx = \frac{b^{ax+b}}{a \cdot \ln b} + c$$

Example

$$\textcircled{1} \int_0^{\ln 2} e^x dx = e^x \Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1 \#$$

$$\textcircled{2} \int 7^{4-3x} dx = \frac{7^{4-3x}}{-3 \cdot \ln 7}$$

$$\textcircled{3} \int e^{6x+2} dx = \frac{e^{6x+2}}{6} + c$$

$$\textcircled{4} \int 2^x e^x \cdot dx = \int (2e)^x dx = \frac{(2e)^x}{1 \cdot \ln 2e} + c$$

$$\frac{(2e)^x}{\ln 2 + \ln e} + c = \frac{(2e)^x}{\ln 2 + 1}$$

$$\boxed{8} \int \frac{1}{x} dx = \ln|x| + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\textcircled{1} \int \frac{4}{x} dx = 4 \ln|x| + c$$

$$\textcircled{2} \int_{-1}^2 \frac{5}{4+x} dx = 5 \ln|4+x| \Big|_{-1}^2 = 5 \ln|6| - 5 \ln|3|$$

$$5(\ln 6 - \ln 3) = 5 \ln \frac{6}{3} = 5 \ln 2 = \ln 32 \#$$

$$(3) \int \frac{2x}{1+x^2} dx = \frac{1}{2} \ln|1+x^2| + C$$

$$(4) \int \tan x dx = \int -\frac{\sin x}{\cos x} dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$(5) \int \frac{x^2}{7+x^3} dx = \frac{1}{3} \ln|7+x^3| + C$$

$$(6) \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

$$(7) \int (\ln e^x + \ln e^{-x}) \cdot dx = \int (x - x) dx$$

$$\int 0 dx = 0 + C = C$$

$$(8) \int \frac{\cos 2x}{1 + \sin 2x} dx = \frac{1}{2} \ln|1 + \sin 2x| + C = 0 + 2 \cos(2x)$$

lecture 6

$$[9] \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{1+(g(x))^2} dx = \tan^{-1}(g(x)) + C$$

Example

$$① \int \frac{1}{1+x^2} dx = \frac{1}{1} \tan^{-1}(x) + C = \tan^{-1}x + C$$

$$② \int \frac{1}{16+x^2} dx = \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + C$$

$$③ \int \frac{1}{5+x^2} dx = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

$$④ \int \frac{3}{1+9x^2} dx = \int \frac{3}{1+(3x)^2} = \tan^{-1}(3x) + C$$

$$⑤ \int \frac{1}{1+16x^2} dx = \int \frac{1}{1+(4x)^2} = \frac{1}{4} \tan^{-1}(4x) + C$$

$$⑥ \int \frac{t}{1+t^4} dt = \int \frac{t}{1+(t^2)^2} dt = \frac{1}{2} \tan^{-1}(t^2) + C$$

$$⑦ \int \frac{x^2}{1+x^6} dx = \int \frac{x^2}{1+(x^3)^2} = \frac{1}{3} \tan^{-1}(x^3) + C$$

$$\boxed{(e^{2x})^2 = e^{4x}} \quad (e^{2x})^2 \neq e^{4x}$$

$$(5) \int \frac{e^{2x}}{1+e^{4x}} dx = \int \frac{e^{2x}}{1+(e^{2x})^2} = \frac{1}{2} \tan^{-1}(e^{2x}) + C$$

$$(6) \int \frac{e^x}{1+e^{4x}} dx = \frac{1}{2} \tan^{-1}(e^x) + C$$

$$(7) \int \frac{t^2-1}{t^4-1} dt = \int \frac{(t-1)(t+1)}{(t+1)(t-1)(t+1)} dt = \int \frac{1}{t^2+1}$$

$$= \tan^{-1}(t) + C$$

lecture 6.7

$$\boxed{10} \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{\sqrt{1-(g(x))^2}} dx = \sin^{-1}(g(x)) + C$$

Example

$$(1) \int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{3^2-x^2}} dx = \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$(2) \int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}} dx = \sin^{-1}(e^{-x}) \Big|_{\ln 2}^{\ln(2/\sqrt{e})}$$

$$= \sin^{-1}(e^{-\ln(2/\sqrt{e})}) - \sin^{-1}(e^{-\ln 2})$$

$$= \sin^{-1}(e^{\ln(2/\sqrt{e})}) - \sin^{-1}(e^{-\ln 2}) = -\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{6}$$

$$\textcircled{3} \int \frac{\sec^2 \theta}{\sqrt{1 - \tan^2 \theta}} d\theta = \int \frac{\sec^2 \theta}{\sqrt{1 - (\tan \theta)^2}}$$

$$\sin^{-1}(\tan x) + C$$

$$\textcircled{4} \int \frac{dx}{x\sqrt{1 - (\ln x)^2}} = \int \frac{\frac{1}{x}}{\sqrt{1 - (\ln x)^2}} dx = \sin^{-1}(\ln x) + C$$

$$\boxed{\text{II}} \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{g(x)\sqrt{g(x)^2 - 1}} dx = \sec^{-1}(g(x)) + C$$

Exampler

$$\textcircled{1} \int \frac{1}{x\sqrt{x^2 - 4}} dx = \frac{1}{2} \sec^{-1}\left(\frac{x}{2}\right) + C$$

$$\textcircled{2} \int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1}(x) + C$$

lecture 68

III Integration by substitution

Example 1

$$\int \frac{\sqrt{1+x^2}}{1+x^2} dx \quad \text{let } y = \tan^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\int \frac{\sqrt{y}}{1+x^2} (1+x^2) dy \quad dx = (1+x^2) dy$$

$$\frac{2y^{3/2}}{3} + C \quad \frac{2(\tan^{-1} x)^{3/2}}{3} + C \quad \#$$

$$\textcircled{2} \int_0^{\pi/2} \sin^2 x \cos x dx \quad \text{let } y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$dx = \frac{dy}{\cos x}$$

$$\int_0^1 y^2 \cos x \frac{dy}{\cos x}$$

$$\frac{y^3}{3} \Big|_0^1 = \frac{1}{3} - \frac{0}{3} = \boxed{\frac{1}{3}} \#$$

$$\text{at } x=0 \quad y \rightarrow \sin(0) = 0$$

$$\text{at } x = \frac{\pi}{2} \quad y \rightarrow \sin\left(\frac{\pi}{2}\right) = 1$$

$$\text{OR} \quad \left(\frac{\sin^3 x}{3} \right) \Big|_0^{\pi/2} = \frac{1}{3}$$

$$\textcircled{3} \int \sqrt{1-x} \, dx$$

$$\begin{aligned} y^2 &= 1-x & x &= 1-y^2 \\ \text{let } y &= \sqrt{1-x} & dy &= \frac{-1}{2\sqrt{1-x}} dx & dy \cdot 2\sqrt{1-x} &= dx \end{aligned}$$

$$\int x \, y \cdot 2\sqrt{1-x} \, dy \rightarrow \int x \, y \cdot (-2) \, y \, dy$$

$$\int (1-y^2)(-2y^2) \, dy = \int (-2y^2 + 2y^4) \, dy$$

$$= \frac{-2}{3} y^3 + \frac{2}{5} y^5 = \frac{-2}{3} (\sqrt{1-x})^3 + \frac{2}{5} (\sqrt{1-x})^5 \Big|_0^1$$

$$\left(\frac{-2}{3} (\sqrt{1-1})^3 + \frac{2}{5} (\sqrt{1-1})^5 \right) - \left(\frac{-2}{3} (\sqrt{1})^3 + \frac{2}{5} (\sqrt{1})^5 \right) =$$

$$\frac{5 \cdot 2}{5 \cdot 3} - \frac{2 \cdot 3}{5 \cdot 3} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

lecture 69

Example

$$\textcircled{1} \int (1+\sqrt{x})^{20} \, dx$$

$$\int y^{20} \cdot 2\sqrt{x} \, dy$$

$$\int y^{20} (2) (y-1)^2 \, dy = \int 2y^{21} - 2y^{20} \, dy$$

$$\frac{2}{22} y^{22} - \frac{2}{21} y^{21} + C \rightarrow \frac{1}{11} (1+\sqrt{x})^{22} - \frac{2}{21} (1+\sqrt{x})^{21} + C$$

$$\begin{aligned} (y-1)^2 &= x \\ \text{let } y &= 1+\sqrt{x} \\ dy &= \frac{1}{2\sqrt{x}} dx \\ 2\sqrt{x} \, dy &= dx \end{aligned}$$

$$(2) \int \frac{\sin x}{16 + 9 \cos^2 x} dx$$

$$\text{let } y = \cos x$$

$$\frac{-1}{\sin x} dy = \frac{-\sin x}{-\sin x} dx$$

$$dx = \frac{-1}{\sin x} dy$$

$$\int \frac{\sin x}{16 + 9y^2} \cdot \frac{-1}{\sin x} dy$$

$$\int \frac{-1}{16 + 9y^2} dy$$

$$= - \int \frac{1}{9(\frac{16}{9} + y^2)} dy$$

$$\int \frac{1}{a^2 + y^2} dy = \frac{1}{a} \tan^{-1}\left(\frac{y}{a}\right) + C$$

$$= - \frac{1}{9} \int \frac{1}{(\frac{4}{3})^2 + y^2} dy$$

$$= \frac{-1}{12}$$

$$= - \frac{1}{9} \cdot \frac{1}{\frac{4}{3}} \tan^{-1}\left(\frac{y}{\frac{4}{3}}\right) + C$$

$$\rightarrow \frac{-3}{(9)(4)} \tan^{-1}\left(\frac{3}{4} \cos x\right) + C$$

$$(3) \text{ If } \int f(x) dx = 10 \text{ find } \int_0^3 x f(x^2 + 1) dx$$

$$\text{let } y = x^2 + 1 \quad \sqrt{y-1} = x$$

$$dy = 2x dx$$

$$dx = \frac{1}{2x} dy$$

$$\text{at } x=0 \quad y=1$$

$$\text{at } x=2 \quad y=5$$

$$\int_0^3 x f(x^2 + 1) dx$$

$$= \frac{1}{2} \int_1^5 f(y) dy$$

$$= \frac{1}{2} (10) = 5$$

H.w

$$\textcircled{1} \int \frac{1}{x \ln x} dx \rightarrow \int \frac{\frac{1}{x}}{\ln x} dx \rightarrow \ln |\ln x| + C$$

$$\text{or } y = \ln x \quad [x dy = dx] \quad \int \frac{1}{x y} \cdot x dy = \ln y + C \\ = \ln |\ln x| + C$$

$$\textcircled{2} \int x^2 e^{x^3} dx \quad \text{let } y = x^3 \quad dy = 3x^2 dx \quad dx = \frac{dy}{3x^2}$$

$$\int x^2 \cdot e^y \frac{dy}{3x^2} = \frac{1}{3} \int e^y dy = \frac{e^{y+1}}{y+1} + C$$

$$\frac{e^{x^3+1}}{x^3+1} + C$$

$$\textcircled{3} \int \frac{\sqrt{1-\frac{1}{x^2}}}{x^2} dx \quad y = \sqrt{1-\frac{1}{x^2}} \quad dy = \frac{\frac{1}{x^2}}{2\sqrt{1-\frac{1}{x^2}}} dx$$

$$2x^2 \sqrt{1-\frac{1}{x^2}} dy = dx \quad \int \frac{y}{x^2} \cdot 2x^2 y dy = \int 2y^2 dy$$

$$\frac{2y^3}{3} + C \rightarrow \frac{2(\sqrt{1-\frac{1}{x^2}})^3}{3} + C$$

$$\textcircled{4} \int \frac{(\ln x)^3}{x} dx \quad y = \ln x \quad x dy = dx$$

$$\int y^3 dy \rightarrow \frac{y^4}{4} + C \rightarrow \frac{(\ln x)^4}{4} + C$$

$$\textcircled{5} \int_{-\ln 3}^{\ln 3} \frac{e^x}{4+e^x} dx \quad \ln |4+e^x| \Big|_{-\ln 3}^{\ln 3} = \left(\ln(4+3) - \ln(3.4 - \frac{1}{3}) \right)$$

$$\ln(7) - \ln\left(\frac{11}{3}\right) \rightarrow \ln\left(\frac{21}{11}\right) \neq$$

$$\textcircled{C} \int \frac{dx}{\sqrt{x}(1+x)} \quad \begin{matrix} y=\sqrt{x} \\ y^2=x \end{matrix} \quad dy = \frac{1}{2\sqrt{x}} dx \quad 2\sqrt{x} dy = dx$$

$$\int \frac{2\sqrt{x}}{\sqrt{x}(1+x)} dy \quad \int \frac{2}{1+y^2} dy \quad 2 \int \frac{1}{1+y^2} dy = 2 \tan^{-1}(y) + C$$

$$2 \tan^{-1}(\sqrt{x}) + C$$

$$\textcircled{F} \int_0^{\ln 3} e^x (1+e^x)^{1/2} dx \quad y = 1+e^x \quad \frac{dy}{e^x} = dx$$

$$= \int y^{1/2} dy \quad \frac{2 y^{3/2}}{3} \rightarrow \frac{2}{3} (1+e^x)^{3/2} \Big|_0^{\ln 3}$$

$$\frac{2}{3} (1+3)^{3/2} - \frac{2}{3} (1+1)^{3/2} = \frac{2}{3} (4^{3/2} - 2^{3/2})$$

$$\frac{2}{3} (\sqrt{64} - \sqrt{8}) = \frac{2}{3} (8 - \sqrt{8}) = \frac{16 - 2\sqrt{8}}{3} \neq$$

$$\textcircled{8} \int x^2 \sqrt{x^3+1} dx \quad y = x^3+1 \quad \frac{dy}{3x^2} = dx$$

$$\frac{1}{3} \int y^{1/2} dy = \frac{2}{3} \frac{y^{3/2}}{3/2} = \frac{2}{9} y^{3/2} \rightarrow \frac{2}{9} (x^3+1)^{3/2} + C$$

$$\textcircled{9} \int x^3 \sin(2x^4) dx \quad y = 2x^4 \quad dy = 8x^3 dx \rightarrow \frac{dy}{8x^3} = dx$$

$$\frac{1}{8} \int \sin(y) dy = -\frac{1}{8} \cos y + C \rightarrow -\frac{1}{8} \cos(2x^4) + C$$

$$(10) \int \sin x e^{\cos x} dx \quad y = \cos x \quad \frac{dy}{- \sin x} = dx$$

$$\int -e^y dx = -e^{\cos x} + C$$

$$(11) \int_{\frac{1}{2}}^{\sqrt{3}/2} \frac{1}{\sin^{-1} x \sqrt{1-x^2}} dx \quad y = \sin^{-1} x \quad dy = \frac{1}{\sqrt{1-x^2}} dx$$

$$\frac{1}{2} \quad \int \frac{1}{y} dy = \ln|y| \rightarrow \ln|\sin^{-1} x| \Big|_{\frac{1}{2}}^{\sqrt{3}/2} \quad \sqrt{1-x^2} dy = dx$$

$$\ln|\sin^{-1} \frac{\sqrt{3}}{2}| - \ln|\sin^{-1} \frac{1}{2}| = \frac{\pi}{3} - \frac{\pi}{6} = \frac{6\pi - 3\pi}{18}$$

$$\frac{3\pi}{18} = \frac{\pi}{6}$$

lecture 70

$$\int \cosh(ax+b) dx = \frac{1}{a} \sinh(ax+b) + c$$

$$\int \sinh(ax+b) dx = \frac{1}{a} \cosh(ax+b) + c$$

$$\int \operatorname{sech}(ax+b) dx = \frac{1}{a} \tanh(ax+b) + c$$

$$\int \operatorname{csch}(ax+b) dx = -\frac{1}{a} \coth(ax+b) + c$$

$$\int \operatorname{sech}(ax+b) \tanh(ax+b) dx = -\frac{1}{a} \operatorname{sech}(ax+b) + c$$

$$\int \operatorname{csch}(ax+b) \coth(ax+b) dx = -\frac{1}{a} \operatorname{csch}(ax+b) + c$$

Example

$$\textcircled{1} \int \sinh(5x+4) dx = \frac{1}{5} \cosh(5x+4) + c$$

$$\textcircled{2} \int \operatorname{sech}(9x) \tanh(9x) dx = -\frac{1}{9} \operatorname{sech}(9x) + c$$

$$\textcircled{3} \int \operatorname{sech}^2(3-9x) dx = \frac{1}{9} \tanh(3-9x) + c$$

$$\textcircled{4} \int \sinh x + \cosh x dx = \cosh x + \sinh x + c = e^x + c$$

$\int e^x dx = e^x$

$$\textcircled{5} \int \frac{\operatorname{sech}(5x) \tanh(5x)}{4 + \operatorname{sech}(5x)} dx \quad \text{let } y = \operatorname{sech}(5x)$$

$$dx = \frac{dy}{-5 \operatorname{sech}(5x) \tanh(5x)}$$

$$-\frac{1}{5} \int \frac{1}{4+y} dy \rightarrow -\frac{1}{5} \ln|4 + \operatorname{sech}(5x)| + c$$

(11) w

$$\textcircled{6} \int \frac{\sinh x}{4 \cosh^3 x + 9} dx \quad y = 4 \cosh^2 x + 9$$

$$dy = 8 \sinh x dx$$

$$\frac{1}{8} \int \frac{1}{y} dy \rightarrow \left\{ \frac{1}{8} \ln |4 \cosh^2 x + 9| \right\} dx = \frac{dy}{8 \sinh x}$$

$$\textcircled{7} \int \sinh^6 x \cosh x dx \quad y = \sinh x$$

$$\frac{dy}{dx} = \cosh x$$

$$\int y^6 dy = \frac{y^7}{7} + C = \frac{\sinh^7 x}{7} + C \quad \#$$

$$\textcircled{14} \frac{d}{dx} \int_a^b f(x) dx = 0 \quad \frac{d}{dx} \int_a^x f(x) dx = f(x)$$

Example

$$\textcircled{1} \text{ find } \frac{d}{dx} \int_0^4 \sqrt{5 + \cos x} dx = 0$$

$$\textcircled{2} \text{ If } y = \int \sqrt{5 + \cos x} \text{ find } \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx} \int \sqrt{5 + \cos x}$$

$$\textcircled{3} \text{ If } \int f(x) dx = x^4 + x + 4 \text{ find } f(x), f'(x)$$

$$\frac{d}{dx} \int f(x) dx = \frac{d}{dx} (x^4 + x + 4)$$

$$f(x) = 4x^3 + 1$$

$$f'(x) = 12x^2 = \boxed{12} \quad \#$$

$$f''(x) = \boxed{5} \quad \#$$

Integration #

(4) If $y = e^{2x} + \ln(x^2+1) + \int_0^y \frac{5x^2}{\cos x} dx$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2e^{2x} + \frac{2x}{x^2+1} + 0 \quad \#$$

(15) $\int f'(x) dx = f(x) + C$
 $\int f''(x) dx = f'(x) + C$
 $\int f'''(x) dx = f''(x) + C$

Example:

(1) If $f'(x) = 5x^4 + 2x + 3$, $f(2) = 3$ find $f(x)$

$$\int f'(x) dx = \int (5x^4 + 2x + 3)$$

$$\ln(x) = x^5 + x^2 + 3x + C = 3 \quad 32 + 4 + 6 + C = 3 \quad \boxed{C = -39}$$

or

$$f(2) + C = x^5 + x^2 + 3x \quad \Rightarrow \quad \boxed{f(x) = x^5 + x^2 + 3x - 39} \quad \#$$

$\text{or } C = 39$

$f(1) = 2$

(2) If $\int f'(x) dx = x^2 + 3x + 3$ $f(1) = 2$ find $f(4)$

$$f(x) + C = x^2 + 3x + 3 \quad \rightarrow \quad f(1) + C = 1^2 + 3 + 3$$

$$2 + C = 7 \quad \boxed{C = 5} \quad f(x) + 5 = x^2 + 3x + 3$$

$$f(x) = x^2 + 3x - 2 \quad \rightarrow \quad f(4) = 4^2 + 3(4) - 2 = 24$$

(3) If $f'(x) = x\sqrt{x^2+1}$ $f(0) = 1$ find $f(x)$?

$$\int f'(x) dx = \int x\sqrt{x^2+1} dx$$

$$f(x) + C = \int x\sqrt{u} \frac{du}{2x}$$

$$f(x) + C = \frac{1}{2} \int u^{1/2} du$$

$$f(x) + C = \frac{1}{2} \cdot \frac{2}{3} u^{3/2}$$

$$f(x) + C = \frac{1}{3} (x^2+1)^{3/2} \quad f(0) = 1$$

$$f(0) + C = \frac{1}{3} (0+1)^{3/2}$$

$$1 + C = \frac{1}{3} \quad \boxed{C = -\frac{2}{3}} \quad \#$$

$$\rightarrow f(x) = \frac{1}{3} (x^2+1)^{3/2} + \frac{2}{3} \quad \#$$

let $u = x^2 + 1$

$$du = 2x dx$$

$$\frac{du}{2x} = dx$$

Homework

(1) If $f'(x) + \cos x = 2x$ $f(\frac{\pi}{2}) = 4$ find $f(x)$

$$f'(x) + \sin x = x^2 + C \quad f(x) = x^2 - \sin x + C$$

$$4 + \frac{1}{\sqrt{2}} = \frac{\pi^2}{4} + C$$

$$C = 4 + \frac{\sqrt{2}}{2} - \frac{\pi^2}{4} = \frac{16}{4} + \frac{2\sqrt{2}}{4} - \frac{\pi^2}{4} = C$$

$$C = \frac{16 + 2\sqrt{2} - \pi^2}{4}$$

$$f(x) = x^2 - \sin x + \frac{16 + 2\sqrt{2} - \pi^2}{4}$$

==

lecture 71

[16] The fundamental Theorem of calculus -

$$\frac{d}{dx} \int_{f(x)}^{g(x)} h(t) dt = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$$

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Example:-

$$\textcircled{1} \frac{d}{dx} \int_1^x \sin t \cdot dt = \sin(x) \cdot 1 - \sin(1) \cdot 0 = \sin x$$

$$\textcircled{2} \frac{d}{dx} \int_{2x}^{x^2} (t^2 - 1) dt = ((x^2)^2 - 1)(2x) - (2x)^2 - 1)2$$

$$2x(x^4 - 1) - 2(4x^2 - 1)$$

$$2x^5 - 2x - 8x^2 + 2$$

$$\textcircled{3} \frac{d}{dx} \int_0^{2x} \sqrt{1+t^4} dt = \sqrt{1+(2x)^4} \cdot 2 - \sqrt{1+0^4} (0) = 2\sqrt{1+16x^4}$$

$$\textcircled{4} \text{ If } g(x) = x^2, \int_6^{3x} \frac{t^5}{t^2+5} dt \text{ Find } g'(2)?$$

$$g'(x) = 2x \int_6^{3x} \frac{t^5}{t^2+5} dt + x^2 \left(\frac{(3x)^5}{(3x)^2+5} \right) (3) + \frac{6^5}{6^2+5} (0)$$

$$2x \int_6^{3x} \frac{t^5}{t^2+5} dt + \frac{3x^2 \cdot (3x)^5}{(3x)^2+5}$$

$$g'(2) = 4 \int_6^6 \frac{t^5}{t^2+5} dt + 3(2)^2 \frac{(3(2))^5}{(3(2))^2+5} = 12 \left(\frac{6^5}{36+5} \right)$$

x is used

2 =

⑤ If $\frac{d}{dx} \int_3^{2x} f(t) dt = \frac{5}{x^2+3}$ Find $f(x)$?

$$\frac{d}{dx} \int_3^{2x} f(t) dt = \frac{-5(2x)}{(x^2+3)^2}$$

$$f(2x) \cdot 2 - f(3) \cdot 0 = \frac{-10x}{(x^2+3)^2} = f(2x) = \frac{-5x}{(x^2+3)^2}$$

$y = 2x$
 $x = \frac{y}{2}$

$$f(y) = \frac{-5(\frac{y}{2})}{((\frac{y}{2})^2+3)^2}$$

$$f(x) = \frac{-5x}{((\frac{x}{2})^2+3)^2}$$

⑥ Find $\lim_{x \rightarrow 1} \frac{\int_2^{2x} (t+3) dt}{x-1}$ عوض القيد = $\frac{0}{0}$
بزوج لقاعدة لوبيتال

$$\lim_{x \rightarrow 1} \frac{(2x+3)(2) - 0}{1} = (2+3)(2) = (5)(2) = 10 \#$$

$$\frac{4x+6}{x-1} \xrightarrow{10/0} \frac{4}{2} = 2$$

Hom

(7) If $g(x) = \int_2^{2x} \frac{t^7}{t+5} dt$ Find $g'(1)$, $g(1)$

$$g(1) = 0$$

$$g'(x) = \frac{(2x)^7 (2)}{(2x)+5} + 0$$

$$g'(x) = \frac{2(2x)^7}{(2x)+5}, \quad g'(1) = \frac{2(2)^7}{2+5} = \frac{256}{7}$$

(8) If $\int_4^{3x} f(y) dy = \frac{-1}{x+1}$ Find $f(x)$

$$f(3x) \cdot 3 = \frac{1}{(x+1)^2}$$

$$f(3x) = \frac{1}{3(x+1)^2}$$

$$y = 3x \rightarrow x = \frac{y}{3}$$

$$f(y) = \frac{1}{3(\frac{y}{3}+1)^2}$$

$$f(x) = \frac{1}{3(\frac{x}{3}+1)^2} \quad \#$$

(9) If $2x + \int_c^x f(t) dt = 8$ Find the value of c .

$$2 + f(x) = 0 \quad (f(x) = -2) \quad -2x + 2c = 8 - 2x$$

$$\int_c^x f(t) dt = 8 - 2x \quad f(t) = -2 \quad -2x \Big|_c^x = 8 - 2x$$

$$2c = 8$$

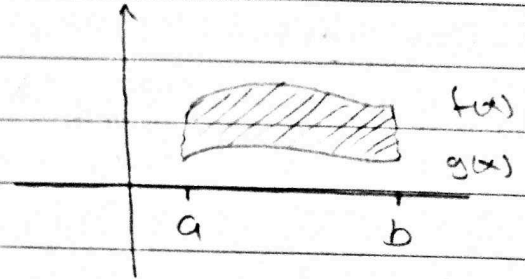
$$\boxed{c = 4}$$

Lecture 72

Area between two curves

Theorem:

$$A = \int_a^b (f(x) - g(x)) dx$$



مثال: $g(x) = f(x)$ if x is between a and b

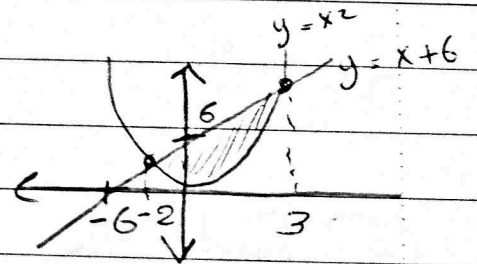
Example:

① find the area of the region enclosed by $y = x^2$ and $y = x + 6$

$$x^2 = x + 6 \rightarrow x^2 - x - 6$$

$$(x - 3)(x + 2)$$

$$x = 3, -2$$



$$A = \int_{-2}^3 ((x+6) - (x^2)) dx$$

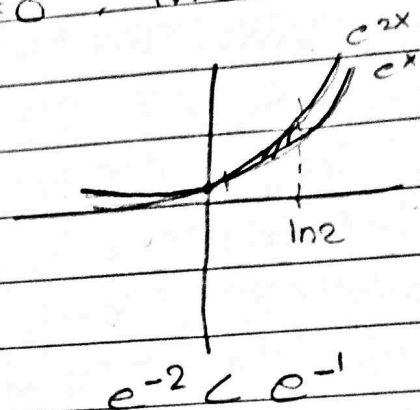
$$\frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^3$$

$$x+6-x^2 \Big|_{-2}^3$$

$$y=0 \quad x=6$$

Q2 find the area bounded $y=e^{2x}$, $y=e^x$
 $x=0$, $\ln 2$

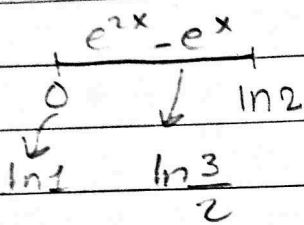
$$A = \int_0^{\ln 2} (e^{2x} - e^x) dx$$



$$e^{2x} = e^x$$

$$2x - x = 0$$

$$x=0$$



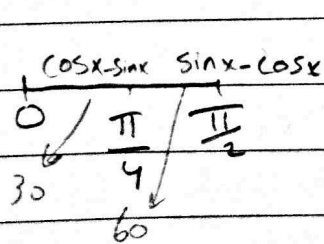
$$\left. \frac{e^{2x}}{2} - e^x \right|_0^{\ln 2}$$

$$\left(\frac{e^{2\ln 2}}{2} - e^{\ln 2} \right) - \left(\frac{e^0}{2} - e^0 \right)$$

$$\frac{3}{2} - \frac{1}{2} = 1$$

$$(2 - 1) - \left(\frac{1}{2} - 1 \right) = +\frac{1}{2}$$

Q3 Find the area enclosed by $y=\sin x$, $y=\cos x$
 $x=0$, $x=\frac{\pi}{2}$? $\sin x = \cos x$



$$\sin 30 = \frac{1}{2}$$

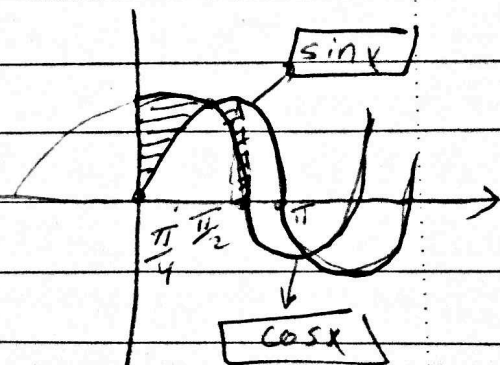
$$\cos 30 = \frac{\sqrt{3}}{2}$$

$$\sin 60 = \frac{\sqrt{3}}{2}$$

$$\cos 60 = \frac{1}{2}$$

$$A = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$\int_0^{\pi/4} (\cos x - \sin x) dx - \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$



Ques 4
 (4) Find the area enclosed by $y = x^3 - 4x$, $y = 0$, $x = -2$, $x = 2$?

$$0 = x^3 - 4x \rightarrow x(x^2 - 4) \quad x = 0, -2, 2$$

$$\begin{array}{c} x^3 - 4x \quad -x^3 + 4x \\ | \quad | \quad | \quad | \\ -2 \quad 0 \quad 2 \end{array}$$

$$\int_{-2}^0 x^3 - 4x \, dx + \int_0^2 -x^3 + 4x \, dx$$

$$\left(\frac{x^4}{4} - 2x^2 \right) \Big|_{-2}^0 + \left(-\frac{x^4}{4} + 2x^2 \right) \Big|_0^2$$

$$0 - (-4) + 4 = 4 + 4 = 8 \#$$

(5) Find the area of the region enclosed by

(a) $y = x^2$, $y = 2x - x^2$

$$x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \quad 2x(x - 1) \quad x = 0, 1$$

$$\begin{array}{c} 2x - x^2 - x^2 \\ | \quad | \\ 0 \quad 1 \end{array} \quad \int_0^1 2x - 2x^2 \, dx \quad \left[x^2 - \frac{2}{3}x^3 \right]_0^1$$

$$\frac{3}{3} - \frac{2}{3} = \frac{1}{3} \#$$

(b) $y = e^x$, $y = x$ on $[0, 1]$

$$\begin{array}{c} e^x - x \\ | \quad | \\ 0 \quad 1 \end{array} \quad \int_0^1 e^x - x \, dx = \left[e^x - \frac{x^2}{2} \right]_0^1$$

$$(e - \frac{1}{2}) - 1 = \boxed{e - \frac{3}{2}} \#$$

⑥ Find the area enclosed by the line $y = x - 1$ and the parabola $y = 2x + 6$

$$x + 1 = \sqrt{2x + 6} = \sqrt{5}$$

$$x + 1 = -\sqrt{2x + 6} = -\sqrt{5}$$

$$\sqrt{2x + 6} = -\sqrt{2x + 6} = -3$$

$$y = (2x + 6)^{1/2}$$

$$y = -(2x + 6)^{1/2}$$

$$\sqrt{2x+6} \quad -\sqrt{2x+6} \quad x-1 \quad -\sqrt{2x+6}$$

-3 -√5 √5

$$(2x+6)^{1/2} - (x-1) \quad | \quad x$$

$$\sqrt{2x+6} = 0$$

$$2x+6 = 0$$

$$x = -3$$

$$\int_{-3}^{-\sqrt{5}} \sqrt{2x+6} + \sqrt{2x+6} \, dx + \int_{-\sqrt{5}}^{\sqrt{5}} x + 1 - \sqrt{2x+6} \, dx$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{1 + (g(x))^2} dx = \tan^{-1}(g(x)) + C$$

##

~~$\int \frac{1}{\sqrt{x^2 + a^2}}$~~

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{\sqrt{1 - (g(x))^2}} dx = \sin^{-1}(g(x)) + C$$

##

$$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{g(x)\sqrt{(g(x))^2 - 1}} dx = \sec^{-1}(g(x)) + C$$

$$\lim_{x \rightarrow 1} \frac{3 + a + a + 3}{0}, \lim_{x \rightarrow 1} \frac{2a + 6}{0} \xrightarrow{\frac{0}{0}} \text{exist} \quad a = -3$$

$2a = 6$
 $a = -3$

$$\lim_{x \rightarrow 1} \frac{3x^2 - 3x - 3 + 3}{x^2 + x - 2}$$

$$\lim_{x \rightarrow 1} \frac{3x(x-1)}{(x+2)(x-1)} = \frac{3}{3} = 1$$

$$a = -3$$

$$\lim_{x \rightarrow 1} f(x) = 1$$

$$C$$