دفتر :

تفاضل و تكامل 2

calculus 2

الاسبوع (7)

للطالب ورد طعمه

اللجنة الأكاديمية لقسم الهندسة الصناعية

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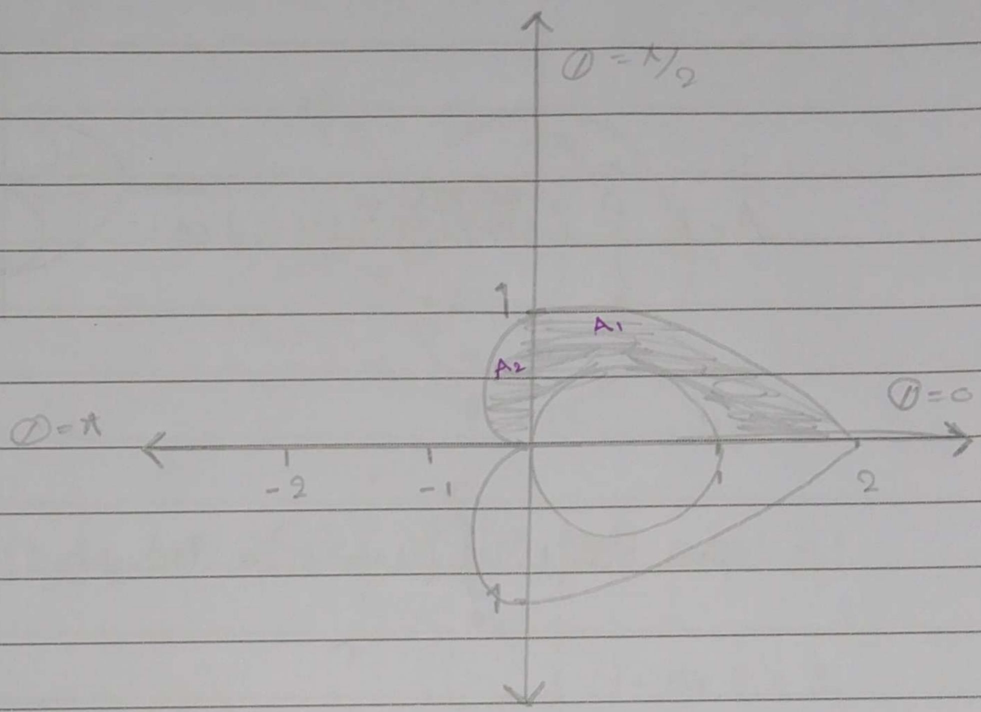
Example:- Find the area of the region in upper half plane outside $r = \cos\theta$ inside $r = 1 + \cos\theta$

$\hookrightarrow 2a = 1 \Rightarrow a = 1/2$

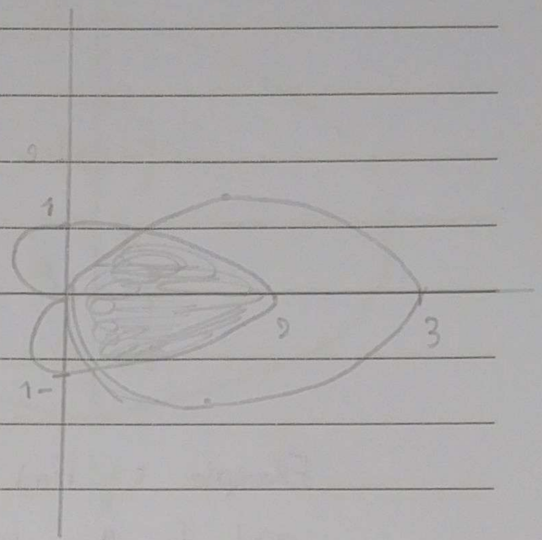


$A = A_1 + A_2$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\cos\theta)^2 - (\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta$$

Example: Find the area of the region that is common to $r = 3\cos\theta$ $r = 1 + \cos\theta$

$\hookrightarrow 2a = 3 \rightarrow a = 3/2$



$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \frac{\pi}{3}, \frac{7\pi}{3}$ $\leftarrow -\frac{\pi}{3}$

$A = A_1 + A_2 + A_3$

$= \frac{1}{2}\int_{-\pi/2}^{-\pi/3} (3\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1+\cos\theta)^2 \, d\theta$

$+ \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2 \, d\theta$

Or

$A = 2\left(\frac{1}{2}\int_{0}^{\pi/3} (1+\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2 \, d\theta\right)$

Example : Find the area enclosed by $r = 4\cos\theta + 2\sin\theta$

$\Downarrow$

$2a = 4 \Rightarrow a = 2$

$2b = 2 \Rightarrow b = 1$

circle with center $(2,1)$ radius $= \sqrt{5}$

$A = \pi r^2$

$= 5\pi$

Example : Find the area in the region that is inside $r=2$ and to the right of $r = \sqrt{2}\sec\theta$

$\sqrt{3} = r\cos\theta \rightarrow x = \sqrt{3}$

vertical line

$2 = \sqrt{2}\sec\theta$

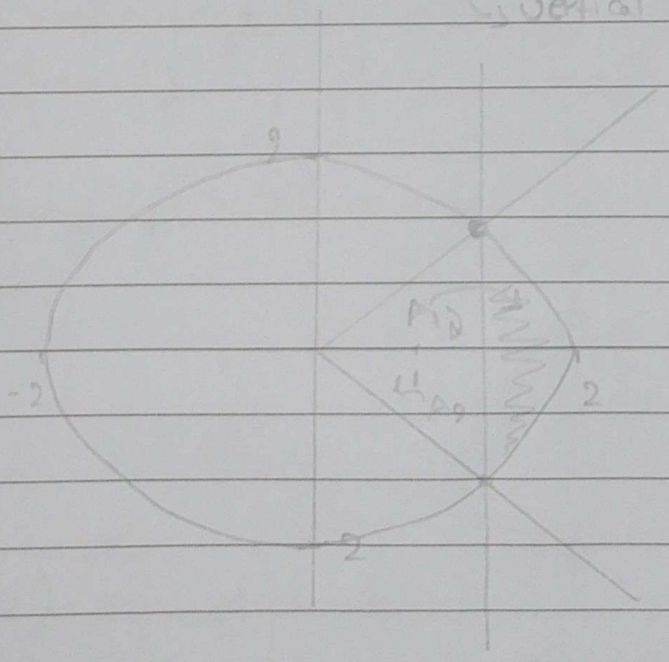
$\sec\theta = \sqrt{2}$

$\theta = \pi/4 , -\pi/4$

نقاط التقاطع

$(2, \pi/4)$

$(-2, \pi/4)$



$$A_1 = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (2)^2\, d\theta = \pi$$

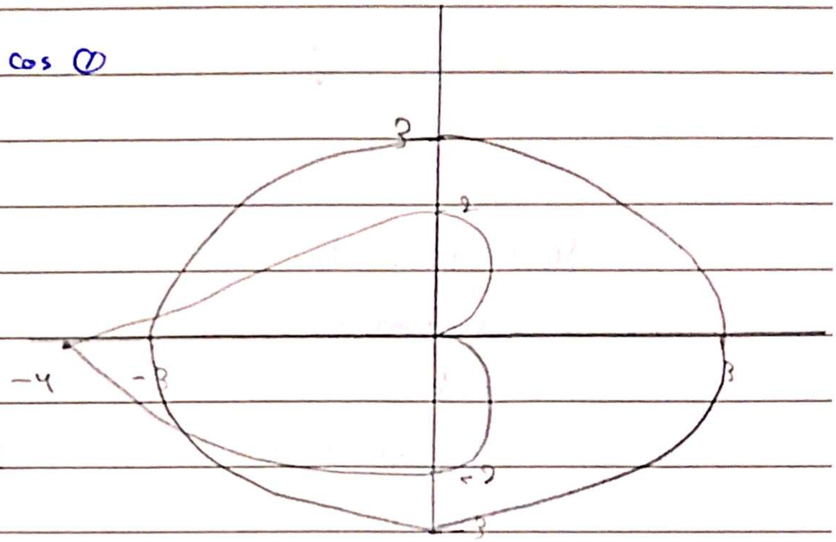
$$A_2 = \frac{1}{2}(2)(2) = 2$$

$$A = A_1 - A_2 = \pi - 2$$

Example : Find all points of intersection :-

[1] $r = 3$

$r = 2 - 2\cos\theta$



$3 = 2 - 2\cos\theta$

$-2\cos\theta = 1$

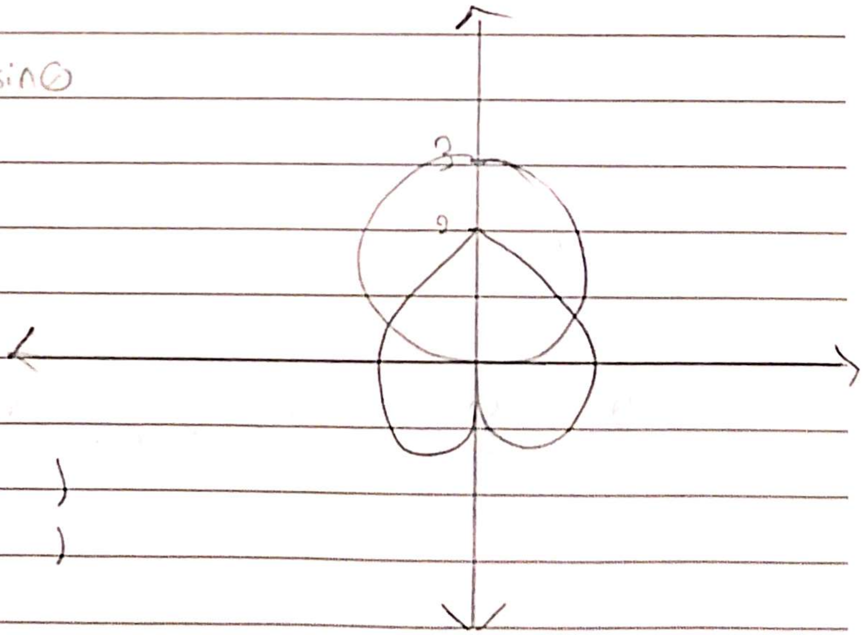
$\cos\theta = -\frac{1}{2}$

$\theta_1 = \frac{2\pi}{3}$, $\theta_2 = \frac{4\pi}{3}$

$\left(3, \frac{2\pi}{3}\right)$ $\left(3, \frac{4\pi}{3}\right)$

[2] $r = 3\sin\theta$

$r = 1 + \sin\theta$



$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$, $\frac{5\pi}{6}$

$\left(\frac{3}{2}, \frac{\pi}{6}\right)$

$\left(\frac{3}{2}, \frac{5\pi}{6}\right)$

pole

Pole : $(0, \theta)$ $\to$ $(0, 0)$, $(0, \pi)$

$\to$ $(0, \frac{\pi}{2})$, $(0, \frac{3\pi}{2})$

المتتابعات - sequences :-

$$\{2, 4, 6, 8, 10, 12, \ldots\}$$

$$\{a_1, a_2, a_3, a_4, \ldots\}$$

first term :- $a_1 = 2$

second term : $a_2 = 4$

⋮

General term : $a_n = f(n) = 2n$

$$\{2n\}_{n=1}^{\infty} \quad , \quad \{2n\}$$

Example : Find the general term of

[1] $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

$a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$, $a_4 = 1/4$

$$a_n = \frac{1}{n} \qquad \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

[2] $\{1/2, 2/3, 3/4, 4/5, 5/6, \ldots\}$

$a_1 = 1/2$, $a_2 = 2/3$, $a_3 = 3/4$, $a_4 = 4/5$

$$a_n = \frac{n}{n+1} \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

[3] $\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\}$

$\quad\;\; 1 \quad 2 \quad 3 \quad 4 \quad 5$

$a_1 = -1 \qquad a_2 = \frac{1}{2} \qquad a_3 = -\frac{1}{3} \qquad a_4 = \frac{1}{4}$

$$a_n = \frac{(-1)^n}{n} \qquad \left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \begin{cases} -\frac{1}{n}, & n \text{ odd} \\ \frac{1}{n}, & n \text{ even} \end{cases}$$

Defintion : A sequence $\{a_n\}$ has the limit L and we write

$\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

If $\lim\limits_{n\to\infty} a_n$ exist we say the sequence converges (or is convergent) otherwise we say the sequence diverges (or is divergent)

* If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is constant then :-

[1] $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

[2] $\lim\limits_{n\to\infty} C a_n = C \lim\limits_{n\to\infty} a_n$

[3] $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

[4] $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$

[5] $\lim\limits_{n\to\infty} (a_n)^p = (\lim\limits_{n\to\infty} a_n)^p$, if $p > 0$ and $a_n > 0$

Example : Find the first four terms and find the limit

[1] $\left\{ \frac{n-1}{2n+1} \right\}_{n=1}^{\infty}$

$a_1 = 0$

$a_2 = 1/5$

$a_3 = 2/7$

$a_4 = 3/9$

Converge to 1/2

$$\lim_{n \to \infty} \frac{n-1}{2n+1} = \frac{1}{2}$$

[2] $\left\{ \frac{2n-1}{3n^2-1} \right\}_{n=1}^{\infty}$

$a_1 = 1/2$

$a_2 = 3/11$

$a_3 = 5/26$

$a_4 = 7/47$

Converge to 0

$$\lim_{n \to \infty} \frac{2n-1}{3n^2-1} = \lim_{n \to \infty} \frac{2}{6n} = 0$$

Example : Determine whether the sequence converge or diverge

[1] $\left\{ 8 - 2n \right\}_{n=1}^{\infty}$

$$\lim_{n \to \infty} (8-2n) = -\infty$$

diverge

[2] $\left\{ \frac{n^2+3n-1}{e^{2n}} \right\}_{n=1}^{\infty}$ ⟶ converge to 0

$$\lim_{n\to\infty} \frac{n^2+3n-1}{e^{2n}} = \lim_{n\to\infty} \frac{2n+3}{2e^{2n}}$$

$$= \lim_{n\to\infty} = \frac{2}{4e^{2n}}$$

$$= 0$$

[3] $\left\{ \tan^{-1}(\sqrt[n]{n}) \right\}_{n=1}^{\infty}$ ⟶ converge to $\pi/4$

$$\lim_{n\to\infty} \tan^{-1}(\sqrt[n]{n})$$

$$\tan^{-1}\left(\lim_{n\to\infty} \sqrt[n]{n}\right) = \tan^{-1}(1)$$

$$= \pi/4$$

$\ast \lim_{n\to\infty} n^{1/n}$

$$y = n^{1/n}$$

$$\ln y = 1/n \ln n$$

$$\lim_{n\to\infty} 1/n \ln n = 0 \cdot \infty$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

$\boxed{e^0 = 1}$

[4] $\left\{ \left(1+\frac{1}{n}\right)^{-n} \right\}_{n=1}^{\infty}$ → converge to $1/e$

Recall

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= e^{1(-1)} = 1/e$$

[5] $\left\{ \left(\frac{n+3}{n+1}\right)^{n} \right\}_{n=1}^{\infty}$ → converge to e^2

$$\lim_{n\to\infty} \left(\frac{n+3}{n+1}\right)^{n} = \lim_{n\to\infty} \left(\frac{\frac{n+3}{n}}{\frac{n+1}{n}}\right)^{n}$$

$$= \lim_{n\to\infty} \left(\frac{1+3/n}{1+1/n}\right)^{n}$$

$$= \lim_{n\to\infty} \frac{(1+3/n)^{n}}{(1+1/n)^{n}}$$

$$= \frac{e^{3(1)}}{e^{1(1)}} = \frac{e^3}{e} = e^2$$

[6] $\left\{ \sqrt{n^2+n+1} - n \right\}_{n=1}^{\infty}$ → converge to $1/2$

$$\lim_{n\to\infty} \left(\sqrt{n^2+n+1} - n\right) * \left(\frac{\sqrt{n^2+n+1} + n}{\sqrt{n^2+n+1} + n}\right)$$

$$\lim_{n\to\infty}\left(\frac{n\,(1+1/n)}{\sqrt{n^2}\;\sqrt{1+1/n+1/n^2}\;+n}\right)$$

$$\lim_{n\to\infty}\left(\frac{\cancel{n}\,(1+1/n)}{\cancel{n}\,(\sqrt{1+1/n+1/n^2}+1)}\right)$$

$$=\frac{1+0}{\sqrt{1+0+0}+1}=\frac{1}{1+1}=1/2$$

Theorem : A sequences converge to L iff even terms converge to L and odd terms converge to L

Examples:- Determine whether the sequence converge

[1] $\left\{ \frac{(-1)^{n+1}}{n} \right\}_{n=1}^{\infty} = \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \right\}$

converge to 0

$$= \begin{cases} -\frac{1}{n} & , n \text{ even} \\ \frac{1}{n} & , n \text{ odd} \end{cases}$$

$$\lim \frac{(-1)^{n+1}}{n} = \begin{cases} \lim\limits_{n\to\infty} -\frac{1}{n} = 0 & , n \text{ even} \\ \lim\limits_{n\to\infty} \frac{1}{n} = 0 & , n \text{ odd} \end{cases} = 0$$

[2] $\left\{ \frac{(-1)^{n+1}}{2n+1} \right\}_{n=1}^{\infty} = \begin{cases} \frac{n}{2n+1} & , n \text{ odd} \\ \frac{-n}{2n+1} & , n \text{ even} \end{cases}$

diverge

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n+1} = \begin{cases} \lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} & , n \text{ odd} \\ \lim\limits_{n\to\infty} \frac{-n}{2n+1} = -\frac{1}{2} & , n \text{ even} \end{cases}$$

Theorem : If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

Example : Determine whether the sequence converge

[1] $\left\{ \frac{(-1)^{n+1}}{n} \right\}_{n=1}^{\infty} = \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \right\}$

converge to 0

$$= \begin{cases} \frac{1}{n} & , n \text{ odd} \\ -\frac{1}{n} & , n \text{ even} \end{cases}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

By theorem $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right| = 0$

Remark : The sequence $\{r^n\}_{n=1}^{\infty}$

[1] Converge to Zero if $-1<r<1$

Ex :- $\{-\frac{1}{2}^n\}_{n=1}^{\infty}$

[2] Converge to 1 if $r=1$

Ex $\{1\}_{n=1}^{\infty}$

Converge $-1<r\leq 1$

[3] Diverge if $r>1$, $r<-1$

Ex $\{(-3)^n\}_{n=1}^{\infty}$

Example: Ditermine whether the sequence converge or diverge:-

[1] $\left\{(-\frac{1}{2})^n\right\}_{n=1}^{\infty}$ Converge to Zero ; $r=-\frac{1}{2}$

[2] $\left\{\frac{1}{3^n}\right\}_{n=1}^{\infty} = \left\{\left(\frac{1}{3}\right)^n\right\}_{n=1}^{\infty}$ $\rightarrow$ Converge to Zero

[3] $\left\{\frac{\pi^n}{5^n}\right\}_{n=1}^{\infty} = \left\{\left(\frac{\pi}{5}\right)^n\right\}_{n=1}^{\infty}$ $\rightarrow$ Converge to Zero

[4] $\left\{ \frac{3^{n+2}}{5^n} \right\}_{n=1}^{\infty}$ $\rightsquigarrow$ converge to Zero

$$\lim_{n\to\infty} \frac{3^{n+2}}{5^n} = 9 \lim_{n\to\infty} \left(\frac{3}{5}\right)^n$$
$$= 9 \ (0)$$
$$= 0$$

[5] $\left\{ \left(\frac{1}{2}\right)^{-n} \right\}_{n=1}^{\infty} = \left\{ 2^n \right\}_{n=1}^{\infty}$ $\rightsquigarrow$ diverge

[6] $\left\{ \frac{1}{2+3^{-n}} \right\}_{n=1}^{\infty} = \lim_{n\to\infty} \frac{1}{2+\left(\frac{1}{3}\right)^n}$

$= \frac{1}{2+0} = 1/2$

converge to $1/2$

Example : Find the Value of a such that $\left\{ \left(\frac{a}{3}\right)^n \right\}_{n=1}^{\infty}$ converge

$\left\{ \left(\frac{a}{3}\right)^n \right\}_{n=1}^{\infty}$ converge if $-1 < \frac{a}{3} \leq 1$

$-3 < a \leq 3$

$a \in (-3, 3]$

* squeezing theorem :-

let a_n, c_n, b_n sequences such that

$a_n \leq c_n \leq b_n$

if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$

the $\lim\limits_{n\to\infty} c_n = L$

Example: Determine whether the sequence converge

$$\left\{ \frac{\sin^2 n}{n} \right\}_{n=1}^{\infty}$$

converge to zero

$$-1 \leq \sin n \leq 1$$

$$0 \leq \sin^2 n \leq 1$$

$$0 \leq \frac{\sin^2 n}{n} \leq \frac{1}{n}$$

$\lim\limits_{n\to\infty} 0 = 0$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

then by theorem

$$\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$$

[2] $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$

$0! = 1$

$1! = 2*1 = 2$

$3! = 3*2*1 = 6$

$\vdots$

$n! = n(n-1)(n-2) \cdots 3*2*1$

$(n+1)! = (n+1)\,n\,(n-1)(n-2)\cdots$

$= (n+1)\,n!$

$$\frac{n(n-1)(n-2)\cdots 3*2*1}{\underbrace{n \cdot n \cdot n \cdot n \cdots n \cdot n}_{n \text{ terms}}}$$

$$0 < \frac{n(n-1)(n-2)\cdots 3*2*1}{n \cdot n \cdot n \cdots n \cdot n} <$$

$$0 < \frac{1}{n}\left(\frac{2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n}{n \cdot n \cdot n \cdots n \cdot n}\right) \leq 1(\frac{1}{n})$$

$$0 < \frac{n!}{n^n} \leq \frac{1}{n}$$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$

by theorm $\lim \frac{n!}{n^n} = 0$

sequences defined recursively :

$a_1 = c$

$a_1, a_2, a_3, a_4, \ldots$

Example : $a_1 = 1$

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$ for all $n \geq 1$

[1] Find a_2, a_3, a_4

$$a_2 = \frac{1}{2}\left(a_1 + \frac{3}{a_1}\right)$$

$$a_2 = \frac{1}{2}\left(1 + \frac{3}{1}\right) = \frac{4}{2} = 2$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{3}{a_2}\right) = \frac{1}{2}\left(2 + \frac{3}{2}\right)$$

$$= \frac{7}{4}$$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{3}{a_3}\right) = \frac{1}{2}\left(\frac{7}{4} + \frac{3 \times 4}{7}\right)$$

$$= \frac{1}{2}\left(\frac{49 + 48}{28}\right)$$

[2] Find the limit of a_n

$$\lim_{n\to\infty} = a_n = \lim_{n\to\infty} = a_{n+1} = \ell$$

$$\lim_{n\to\infty} = a_n = \lim_{n\to\infty} a_{n+1}$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$$

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{3}{a_n}\right)$$

$$\ell = \frac{1}{2}\left(\ell + \frac{3}{\ell}\right)$$

$$2\ell = \ell + \frac{3}{\ell}$$

$$\ell = \frac{3}{\ell}$$

$$\ell^2 = 3 \Rightarrow \ell = \pm\sqrt{3}$$

$$\therefore \ell = +\sqrt{3}$$

ننظر لسلوك الأعداد والمعادلة المعطاة

Ex:- 1, 2, 1, 2, 1, 2, ...

$L = 3/2$ ✗

diverge

Definition: A sequence $\{a_n\}$ is called

Increasing if $a_1 < a_2 < a_3 \cdots a_n < a_{n+1} < \cdots$
$(a_n < a_{n+1}$ for all $n \geq 1)$

Decreasing if $a_1 > a_2 > a_3 \cdots a_n > a_{n+1} > \cdots$
$(a_n > a_{n+1}$ for all $n \geq 1)$

A sequence is monotonic if it is increasing or decreasing

Testing for monotonicity :-

	Increasing $a_{n+1} > a_n$	Decreasing $a_{n+1} < a_n$
[1] Difference الفرق	$a_{n+1} - a_n > 0$	$a_{n+1} - a_n < 0$
حدود موجبة !! → [2] Ratio النسبة	$\frac{a_{n+1}}{a_n} > 1$	$\frac{a_{n+1}}{a_n} < 1$
[3] Diffrentiation الاشتقاق	$a_n = f(x)$ $f'(x) > 0$	$a_n = f(x)$ $f'(x) < 0$

Example: Determine wether the sequence monotonic

[1] $\left\{ 1 - \frac{1}{n} \right\}_{n=1}^{\infty}$

* $a_n = 1 - \frac{1}{n}$ $\qquad a_{n+1} = 1 - \frac{1}{n+1}$

$a_{n+1} - a_n = \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right)$

increasing $\forall n \geq 1$

$= \cancel{1} - \frac{1}{n+1} - \cancel{1} + \frac{1}{n}$

$= \frac{1}{n(n+1)} > 0$

* $\frac{a_{n+1}}{a_n} = \frac{1 - \frac{1}{n+1}}{1 - \frac{1}{n}}$

$= \frac{\frac{n+1-1}{n+1}}{\frac{n-1}{n}} = \frac{n}{n+1} * \frac{n}{n-1}$

$= \frac{n^2}{n^2-1} > 1$

* $a_n = f(x) = 1 - \frac{1}{x}$

$f'(x) = 0 + \frac{1}{x^2}$ $\qquad \overset{+}{\longleftarrow\!\!\!\underset{0}{|}\!\!\!\longrightarrow} f'$

$f'(x) > 0$

[2] $\left\{ \frac{10^n}{n!} \right\}_{n=1}^{\infty}$ $a_n = \frac{10^n}{n!}$

$a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

decreasing

$\forall n \geq 10$

$\forall n > 9$

$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$

$= \frac{\cancel{10^n} \cdot 10 \quad \cancel{n!}}{(n+1)\, \cancel{n!} \quad \cancel{10^n}}$

$= \frac{10}{n+1} < 1 \quad \forall n \geq 9$

$(a_1 \sim a_9)$ متزايد $(a_{11} \longrightarrow \infty)$ تناقص

[3] $\{ n^5 e^{-n} \}_{n=1}^{\infty}$

$a_n = f(x) = x^5 e^{-x}$

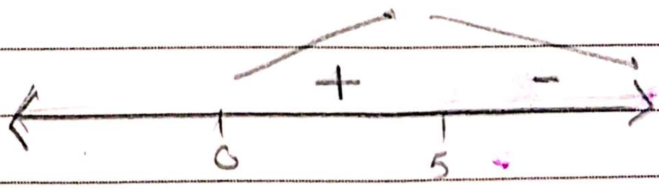
$f'(x) = 5x^4 e^{-x} + x^5 (-e^{-x}) = 0$

decreasing

$\forall n \geq 5$

$x^4 e^{-x} (5-x) = 0$

$x = 0$ $e^{-x} \neq 0$ $x = 5$



f increasing $(0,5)$

f decreasing $(5,\infty)$

** ملاحظة lecture 52 غير مطلوب